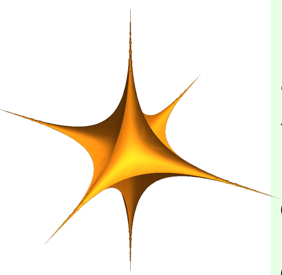
Homework 10: Functions

This homework is due Friday, 10/5/2019. The topic of function reviews also a bit of vector-valued functions like curves $\vec{r}(t)$ and parametrized surfaces $\vec{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$.

1 For which functions is the domain of f the entire plane?
a) $f(x,y) = ye^{1/x}$
b) $f(x,y) = \log(1 + |x+y|)$
c) $f(x,y) = \sqrt{(}|x-y|)$
d) $f(x,y) = e^x/(1+\sin(y))$



Solution:
a) No, this is not defined for $x = 0$.
b) yes
c) yes
d) no, this is not defined for $y = 3\pi/2$.

2 When defining parametrized surfaces $\vec{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$ we use functions of two variables in all coordinates. Coming up with good parametrizations can be a bit tricky sometimes. We look at the surface S given as a level surface $|x|^{2/5} + |y|^{2/5} + |z|^{2/5} = 1$. Our goal is to parametrize S. To do so, start with the case when $2/5$ is replaced by 2. Then modify the parametrization by changing each function in a suitable way. All functions $x(u,v)$, $y(u,v)$ and $z(u,v)$ will be nice and differentiable (we talk about derivatives in the next hour).

Solution:
Start with $\vec{r}(\theta,\phi) = [\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)]$. In order to compensate for the $1/5$, we just take the 5th power in each coordinate $\vec{r}(\theta,\phi) = [\cos^5(\theta)\sin^5(\phi), \sin^5(\theta)\sin^5(\phi), \cos^5(\phi)]$.

3 If $\vec{r}(t)$ is a parametrization, the resulting "curve" is defined as all the values of $\vec{r}(t)$, when t is a real number $t \in \mathbb{R}$. Look at the range of $\vec{r}$ in the following two cases?
a) $\vec{r}(t) = [\cos(t), \sin(2t)]$?
b) $\vec{r}(t) = [\cos(t), \sin(\sqrt{2}t)]$
The "curves" in the two examples look different. Why? To investigate, you might want to plot the two curves with a computer for t in an interval like $[0, 200\pi]$.

Solution:
Note that for functions $r : \mathbb{R} \to \mathbb{R}^2$, the range is the curve you draw when following $r(t)$. It was for functions $\mathbb{R} \to \mathbb{R}$ where the range was part of the real line, usually an interval. Now the range is part of the plane. For a curve with finite interval as domain, the range is the curve drawn. If we draw the curve with $t \in \mathbb{R}$, we can get a curve or we can get more complicated shapes. In this example, we see in b) that this shape can be a square.
a) In the first case the curve is closed. The range is a figure 8 curve.
b) In the second case, the curve covers the entire square $[-1, 1] \times [-1, 1]$ as $\sqrt{2}$ is irrational. For $\vec{r}(t) = [\cos(t), \sin((p/q)t)]$ we would always have get a closed curve as a range. But $\sqrt{2}$ can not be written as a fraction.

4 Consider the function $f(x, y) = x/\sqrt{x^2 + y^2}$. It has the property that the two level curves $f = 0$ and $f = 1$ intersect at $(0, 0)$.
a) Draw $f = 0$ and parametrize it as $\vec{r}_1(t)$ so that $\vec{r}_1(0) = [0, 0]$.
b) Draw $f = 1$ and parametrize it as $\vec{r}_2(t)$ so that $\vec{r}_2(0) = [0, 0]$.
c) Is there a value $a = f(0, 0)$ so that $t \to f(\vec{r}_1(t))$ and $t \to$

$f(\vec{r}_2(t))$ are both continuous functions in t with the requirement $f(\vec{r}_1(0)) = f(\vec{r}_2(t) = a$? If yes, what is the value a? If no, why is there no value?

Solution:
a) For $x = 0$, the curve is $x = 0$ which is parametrized by $r_1(t) = [0, t]$.
b) For $y = 0$, the curve is $y = 0$ which is parametrized by $r_2(t) = [t, 0]$. c) There is no value a as following the curve $r_1(t)$ we would be forced to have $a = 0$. When following the curve $r_2(t)$ we would be forced to have $a = 1$.

5 If we want to investigate whether the limiting value of $f(x, y)$ at $(0, 0)$ is defined, one strategy is to use polar coordinates and check whether $\lim_{r\to 0} f(r\cos(\theta), r\sin(\theta))$ is defined. In the previous problem you have seen that there was no way to define $f(0, 0)$ to make f continuous. The problem can be that approaching $(0, 0)$ along curves from different directions gives different values. Decide in the following three cases, a limiting value $\lim_{(x,y)\to(0,0)} f(x, y)$ exists. You might want to recall the l'Hôpital rule.
a) $f(x, y) = \frac{\cos(x^2+y^2)-1}{x^2+y^2}$ if $(x, y) \neq (0, 0)$ and $f(x, y) = 1/2$ if $(x, y) = 0$.
b) $f(x, y) = \frac{x^6-y^6}{(x^2+y^2)^2}$ if $(x, y) \neq (0, 0)$ and $f(x, y) = 0$ if $(x, y) = 0$.
c) $f(x, y) = \frac{x^2-y^2}{(x^2+y^2)}$ if $(x, y) \neq (0, 0)$ and $f(x, y) = 0$ if $(x, y) = 0$.

Solution:

a) In polar coordinates, we have $\cos(r^2) - 1)/r^2$ which has by Hopital the limit 0 for $r = 0$. The function is continuous if the function value $f(0,0) = 0$ is assumed. But as the function value has been put to $1/2$, the function is not continuous. b) In Polar coordinates, the function is $r^2(\cos^2(\theta) - \sin^2(\theta))$. As $r \to 0$, this gives zero. The function is continuous.

c) In Polar coordinates, we have $\cos^2(\theta) - \sin^2(\theta)$ which depends on θ. For $\theta = 0$ for example, the value is 1. The function is 1 on the positive x-axes and does not converge for $r \to 0$ to the value 0 which was assigned. The limit depends on the angle at which the origin is approached.

Main points

The **domain** of a function or parametrization is the place where the function is defined. Notions of continuity for functions are similar in the multi-variable case: for example, a function $f(x, y)$ with domain R is **continuous at** $(a, b) \in R$ if $f(x, y) \to f(a, b)$ for all choices $(x, y) \to (a, b)$. To decide about continuity of $f(x, y)$, polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$ can help. For $f(x, y) = \frac{x^2y^2}{x^2+y^2}$ for example, we have $f(r, \theta) = \cos^2(\theta) - \sin^2(\theta)$. As arbitrarily close to $(0, 0)$, the function takes any value in $[-1, 1]$, the function is not continuous. For vector valued functions like parametrizations of surfaces $\vec{r}(u, v) = [x(u, v), y(u, v), z(u, v)]$, the domain is the intersection of the domains for the three functions.