Homework 6: Arc length

This homework is due Monday, 9/23. As you all might have felt a bit rushed in unit 3 and curvature was removed from unit 6, we start with two review problems before doing arc length.

1 Let L be the line parametrically by $\vec{r}(t) = [1 + 2t, 4 + t, 2 + 3t]$ and M be the line through the points $P = (-5, 2, -3)$ and $Q = (1, 2, -6)$.
a) The lines L and M intersect; find the point of intersection.
b) How many planes contain both lines?
c) Give a parametric equation for a plane Π that contains both lines.

2 A group of students is asked to find the distance between a given point P and a given line L in $\mathbb{R}^3$. Below are examples of student strategies. Your job is to decide which strategies work and which do not. Sketch a picture that illustrates each strategy and explain in a sentence or two.
a) Barry suggests we pick any point Q on the line L and find the distance from P to Q.
b) Olivia suggests we find any plane Σ perpendicular to L that passes through P. Then we find the point R where L and Σ intersect and calculate the distance from R to P.
c) Lynn suggests we start by taking any two points S and T on the line L. Then we find the area a of the triangle $\triangle SPT$. Finally, thinking of the line segment ST as the base of the triangle and calling its length b, we solve for the height of the triangle h in the equation $a = \frac{1}{2}bh$. Then h will be the distance between L and P.
d) Billy suggests we take any vector $\vec{v}$ perpendicular to the line L, then find a scalar t so that $P + t\vec{v}$ is a point on the line L. Then $|t|$ will be the distance from P to L.

3 a) Find the arc length of the curve $\vec{r}(t) = [3\cos(t^5), 3\sin(t^5), 3t^5]$, where $-1 \leq t \leq 1$.
b) Compute the arc length of $\vec{r}(t) = [t^3, 24t, 6t^2]$ with $t \in [0, 6]$.

4 Arc length can be defined in any dimensions. A curve in 4 dimension is parametrized as $\vec{r}(t) = [x_1(t), x_2(t), x_3(t), x_4(t)]$. Find the arc length of $\vec{r}(t) = [t, \log(t), 1/t, \log(t)]$ where $\log(t) = \ln(t)$ is the natural log and $1 \leq t \leq 4$.

5 a) Use a calculator, Mathematica or Wolfram alpha to evaluate the arc length of the curve $\vec{r}(t) = [\mathrm{Cos}[t], \mathrm{Sin}[t], t^4]$ from $t = 0$ to $t = 9$.
b) Do the same with $\vec{r}(t) = [\mathrm{Cos}[t^2], \mathrm{Sin}[t^2], t^8]$ from $t = 0$ to $t = 3$. Compare with the result in a) and explain you got the same result.

Definitions

If $t \in [a, b] \mapsto \vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$ and speed $|\vec{r}\,'(t)|$, then

$$\int_a^b |\vec{r}\,'(t)| \, dt$$

is called the **arc length of the curve**. Written out in coordinates for example, $\vec{r}(t) = [x(t), y(t), z(t)]$, we have

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \, dt \, .$$