Homework 6: Arc length

This homework is due Monday, 9/23. As you all might have felt a bit rushed in unit 3 and curvature was removed from unit 6, we start with two review problems before doing arc length.

1 Let L be the line parametrically by $\vec{r}(t) = [1+2t, 4+t, 2+3t]$ and M be the line through the points $P = (-5, 2, -3)$ and $Q = (1, 2, -6)$.
a) The lines L and M intersect; find the point of intersection.
b) How many planes contain both lines?
c) Give a parametric equation for a plane Π that contains both lines.

Solution:
a) The line M is parametrized as $[-5, 2, -3] + t[6, 0, -3]$. As in the previous homework, we have to find a common solution to

$$\vec{r}_1(t) = [1+2t, 4+t, 2+3t]$$

and

$$\vec{r}_2(s) = [-5+6s, 2, -3-3s] \ .$$

There is a solution at $t = -2, s = 1/3$, so that $P = (-3, 2, -4)$ is a point of intersection.
b) There is exactly one plane containing both lines. If the two lines would have been the same, then we would have had infinitely many.
c) One possible parametrisation of this plane is $\vec{r}(t, s) = [-3, 2, -4] + t[6, 0, -3] + s[2, 1, 3]$.

2 A group of students is asked to find the distance between a given point P and a given line L in $\mathbb{R}^3$. Below are examples of student

strategies. Your job is to decide which strategies work and which do not. Sketch a picture that illustrates each strategy and explain in a sentence or two.

a) Barry suggests we pick any point Q on the line L and find the distance from P to Q.

b) Olivia suggests we find any plane Σ perpendicular to L that passes through P. Then we find the point R where L and Σ intersect and calculate the distance from R to P.

c) Lynn suggests we start by taking any two points S and T on the line L. Then we find the area a of the triangle $\triangle SPT$. Finally, thinking of the line segment ST as the base of the triangle and calling its length b, we solve for the height of the triangle h in the equation $a = \frac{1}{2}bh$. Then h will be the distance between L and P.

d) Billy suggests we take any vector $\vec{v}$ perpendicular to the line L, then find a scalar t so that $P + t\vec{v}$ is a point on the line L. Then $|t|$ will be the distance from P to L.

Solution:

a) This strategy does not work. Taking a general point P on L and a general point Q on M gives in general larger distances. There is just one choice of P and Q (if the lines are not parallel) for which the distance between P and Q is the distance between the lines. But what is important is that the connection between P and Q is perpendicular to both lines.

b) This strategy does work. The intersection point of Σ with L gives a point R on the line for which the connection segment PR is perpendicular to L. This is gives indeed the right distance.

c) Also this strategy works. This is a good way to think about the problem. A bit more efficient even is to look at the parallelogram and note that the parallelogram area divided by the base length is the height of the parallelogram. The advantage of the later approach is that one does not have to worry about the factor 2.

d) This strategy does not work. Billy would have to be very lucky to find the right vector v so that $P + t\vec{v}$ is on the line.

3 a) Find the arc length of the curve $\vec{r}(t) = [3\cos(t^5), 3\sin(t^5), 3t^5]$, where $-1 \leq t \leq 1$.

b) Compute the arc length of $\vec{r}(t) = [t^3, 24t, 6t^2]$ with $t \in [0, 6]$.

Solution:

a) We have $\vec{r}\,'(t) = 15t^4[\sin(t^4), \cos(t^4), 1]$ and $|\vec{r}\,'(t)| = 15\sqrt{2}t^4$. Integrate, to get $6\sqrt{2}$.

b) We have $|\vec{r}(t)| = 3(8 + t^2)$. Integrating this over the interval $[0, 6]$ gives 360.

4 Arc length can be defined in any dimensions. A curve in 4 dimension is parametrized as $\vec{r}(t) = [x_1(t), x_2(t), x_3(t), x_4(t)]$. Find the arc length of $\vec{r}(t) = [t, \log(t), 1/t, \log(t)]$ where $\log(t) = \ln(t)$ is the natural log and $1 \leq t \leq 4$.

Solution:
The velocity is $\vec{r}\,'(t) = [1, 1/t, -1/t^2, 1/t]$. Its length gives the speed $|\vec{r}\,'(t)| = \sqrt{1 - 2/t^2 + 1/t^4}$. This expression factors. We have to integrate $(1 + t^2)/t^2$ from $t = 1$ to $t = 4$. The answer is 15/4.

5 a) Use a calculator, Mathematica or Wolfram alpha to evaluate the arc length of the curve $\vec{r}(t) = [\mathrm{Cos}[t], \mathrm{Sin}[t], t^4]$ from $t = 0$ to $t = 9$.
b) Do the same with $\vec{r}(t) = [\mathrm{Cos}[t^2], \mathrm{Sin}[t^2], t^8]$ from $t = 0$ to $t = 3$. Compare with the result in a) and explain you got the same result.

Solution:
a) The arc length equals to $\int_0^9 \sqrt{1 + 16t^6}\ dt$. Numerical integration gives us ~ 6561.66.
b) the result is the same because we have a different parametrization. Take the time $s = t^2$ in the curve of a).

Definitions

If $t \in [a,b] \mapsto \vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$ and speed $|\vec{r}\,'(t)|$, then

$$\int_a^b |\vec{r}\,'(t)| \; dt$$

is called the **arc length of the curve**. Written out in coordinates for example, $\vec{r}(t) = [x(t), y(t), z(t)]$, we have

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \; dt \; .$$