Homework 3: Cross product, lines, planes

This homework is due Monday, 9/16/2019

1 a) What is the equation of the plane containing $P = (2, 2, 2), Q = (5, 7, 4), R = (6, 6, 1)$. b) Find the area of the triangle PQR.

2 We are given four points $A = (2, 2, 3), B = (4, 0, 7), C = (6, 3, 1)$ and $D = (2, -3, 11)$.
a) Find the distance $d(D, \Sigma)$ between D to the plane Σ through A, B, C. Are the four points in a common plane?
b) Find the distance $d(D, L)$ of D to the line L through AB.
c) Given 4 arbitrary points defining L and Σ as in the example. Is there a relation between $d(D, L)$ and $d(D, \Sigma)$? (One of them is always larger or equal than the other? Which one? No computations are needed.)

3 a) Find an equation of the plane containing the line of intersection of the planes $x - z = 1$ and $y + z = 3$ which is perpendicular to the plane $x + y - 2z = 1$.
b) Find the distance of the plane found in a) to the origin $(0, 0, 0)$.

4 a) Parametrize the line L through $P = (2, 1, 2)$ that intersects the line $x = 1 + t, y = 1 - t, z = 2t$ perpendicularly.
b) Parametrize the y-axis.
c) What is the distance from this line L to the y-axis?

5 To compute the distance between a plane $ax + by + cz + dw = e$ in four dimensional space and a point P, we can use the known formula $|\vec{PQ} \cdot \vec{n}|/|\vec{n}|$ familiar in three dimensional space. Its just that the vector $\vec{n} = [a, b, c, d]$ has now four coordinates and Q is a point on the plane. Find the distance between the plane $x + 3y + 5z + w = 1$ to the point $P = (1, 1, 1, 1)$.

Main definitions

The **cross product** of two vectors $\vec{v} = [v_1, v_2, v_3]$ and $\vec{w} = [w_1, w_2, w_3]$ in space is defined as the vector

$$\vec{v} \times \vec{w} = [v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1] \, .$$

The number $|\vec{v} \times \vec{w}|$ defines the **area of the parallelogram** spanned by $\vec{v}$ and $\vec{w}$. It satisfies $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$.

The scalar $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is called the **triple scalar product** of $\vec{u}, \vec{v}, \vec{w}$. The number $|[\vec{u}, \vec{v}, \vec{w}]|$ defines the **volume of the parallelepiped** spanned by $\vec{u}, \vec{v}, \vec{w}$. The **orientation** given by the sign of $[\vec{u}, \vec{v}, \vec{w}]$.

A point $P = (p, q, r)$ and a vector $\vec{v} = [a, b, c]$ define the **line** $L = \{[x, y, z] = [p, q, r] + t[a, b, c], t \in \mathbb{R} \}$.

A point P and two vectors $\vec{v}, \vec{w}$ define a **plane** $\Sigma = \{\vec{OP} + t\vec{v} + s\vec{w}$, where t, s are real numbers $\}$.
An example is $\Sigma = \{[x, y, z] = [1, 1, 2] + t[2, 4, 6] + s[1, 0, -1] \ \}$. This is called the **parametric description** of a plane. The implicit equation of the plane $\vec{x} = \vec{x}_0 + t\vec{v} + s\vec{w}$ is $ax + by + cz = d$, where $[a, b, c] = \vec{v} \times \vec{w}$ is a vector normal to the plane and d is obtained by plugging in $\vec{x}_0$. For distances, there is an extra handout.