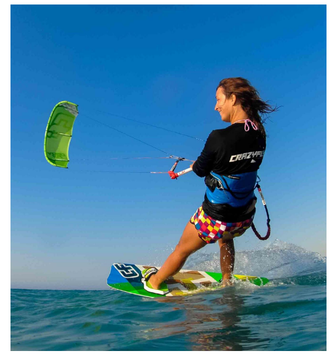
Homework 2: Vectors and Dot product

This homework is due on Friday, 9/13 at the beginning of class.



1 A **kite surfer** gets pulled with a force $\vec{F} = [7, 1, 4]$. She moves with velocity $\vec{v} = [4, -2, 1]$. The dot product of $\vec{F}$ with $\vec{v}$ is **power**.
a) What is the angle between the $\vec{F}$ and $\vec{v}$?
b) Find the **vector projection** of the $\vec{F}$ onto $\vec{v}$.

2 Light shines long the vector $\vec{a} = [a_1, a_2, a_3]$ and reflects at the three coordinate planes where the angle of incidence equals the angle of reflection. Verify that the reflected ray is $-\vec{a}$. **Hint**. Reflect first at the xy-plane. What happens with the vector $\vec{a}$?

3 a) In order to see whether two data points $\vec{v} = [1, 1, -2]$ and $\vec{w} = [1, -2, 1]$ are correlated, we compute the cosine of the angle between the two vectors. Do this for the vectors $\vec{v}$ and $\vec{w}$.
b) Find two vectors $\vec{a}$ and $\vec{b}$ for which all coordinates are positive such that the angle between them is $\pi/4 = 45°$. In statistics the dot product between $\vec{v}$ and $\vec{w}$ is also called the **covariance** and the lengths $|\vec{v}|$ and $|\vec{w}|$ are called the **standard deviations** of $\vec{v}$ and $\vec{w}$. A data scientist calls the cosine of the angle the **correlation**.

4 a) Find the angle between a space diagonal of a cube and the diagonal in one of its faces.
b) The **hypercube** is also called the **tesseract**. It has vertices $(\pm 1, \pm 1, \pm 1, \pm 1)$. Find the angle between the hyper diagonal connecting $(1, 1, 1, 1)$ with $(-1, -1, -1, -1)$ and the space diagonal connecting $(1, 1, 1, 1)$ with $(-1, -1, -1, 1)$.

5 a) Verify that if $\vec{a}, \vec{b}$ are nonzero vectors, then $\vec{c} = |\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ bisects the angle between $\vec{a}, \vec{b}$ if $\vec{c}$ is not zero.

b) Verify the parallelogram law $|\vec{a}+\vec{b}|^2+|\vec{a}-\vec{b}|^2 = 2|\vec{a}|^2+2|\vec{b}|^2$.

Main definitions

Two points $P = (a, b, c)$ and $Q = (x, y, z)$ define a **vector** $\vec{v} = [x - a, y - b, z - c]$. We also write $\vec{v} = \vec{PQ}$. The numbers v_1, v_2, v_3 in $\vec{v} = [v_1, v_2, v_3]$ are the **components** of $\vec{v}$. The **length** $|\vec{v}|$ of a vector $\vec{v} = \vec{PQ}$ is defined as the distance $d(P, Q)$ from P to Q. A vector of length 1 is called a **unit vector**. The **addition** is $\vec{u} + \vec{v} = [u_1, u_2, u_3] + [v_1, v_2, v_3] = [u_1+v_1, u_2+v_2, u_3+v_3]$. The **scalar multiple** $\lambda\vec{u} = \lambda[u_1, u_2, u_3] = [\lambda u_1, \lambda u_2, \lambda u_3]$. The difference $\vec{u} - \vec{v}$ can be seen as $\vec{u} + (-\vec{v})$.

The **dot product** of two vectors $\vec{v} = [a, b, c]$ and $\vec{w} = [p, q, r]$ is defined as $\vec{v} \cdot \vec{w} = ap + bq + cr$. The **Cauchy-Schwarz inequality** tells $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$.

The **angle** between two nonzero vectors is defined as the unique $\alpha \in [0, \pi]$ satisfying $\vec{v} \cdot \vec{w} = |\vec{v}| \; |\vec{w}| \cos(\alpha)$. Two vectors are called **orthogonal** or **perpendicular** if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. For example, $\vec{v} = [2, 3]$ is orthogonal to $\vec{w} = [-3, 2]$. The vector $\mathrm{P}(\vec{v}) = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$. The **scalar projection** $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}$ is plus or minus the length of the projection of $\vec{v}$ onto $\vec{w}$. The vector $\vec{b} = \vec{v} - P(\vec{v})$ is a vector orthogonal to $\vec{w}$. **Pythagoras tells:** if $\vec{v}$ and $\vec{w}$ are orthogonal, then $|v - w|^2 = |v|^2 + |w|^2$.