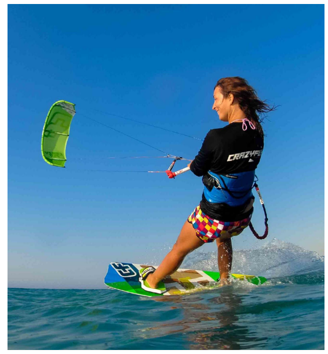
Homework 2: Vectors and Dot product

This homework is due on Friday, 9/13 at the beginning of class.



1 A **kite surfer** gets pulled with a force $\vec{F} = [7, 1, 4]$. She moves with velocity $\vec{v} = [4, -2, 1]$. The dot product of $\vec{F}$ with $\vec{v}$ is **power**.
a) What is the angle between the $\vec{F}$ and $\vec{v}$?
b) Find the **vector projection** of the $\vec{F}$ onto $\vec{v}$.

Solution:
(a) To find the angle between the force and velocity, we use the formula:

$$\cos\theta = \frac{\vec{F}\cdot\vec{v}}{|\vec{F}||\vec{v}|}.$$

The magnitude of the force is $\sqrt{7^2 + 1^2 + 4^2} = \sqrt{66}$. The magnitude of the velocity is $\sqrt{21}$. The dot product $\vec{F}\cdot\vec{v} = 30$. Thus,

$$\cos\theta = \frac{F\cdot v}{|F||v|} = \frac{30}{\sqrt{66}\cdot\sqrt{21}}$$

Hence, $\theta = \arccos(\frac{30}{\sqrt{1386}})$.
(b) The projection of $\vec{F}$ onto $\vec{v}$ is given by

$$\text{proj}_v F = \frac{F\cdot v}{|v|^2}v = \frac{30}{\sqrt{21}^2}[4, -2, 1] = \frac{10}{7}[4, -2, 1].$$

2 Light shines long the vector $\vec{a} = [a_1, a_2, a_3]$ and reflects at the three coordinate planes where the angle of incidence equals the

angle of reflection. Verify that the reflected ray is $-\vec{a}$. **Hint**. Reflect first at the xy-plane. What happens with the vector $\vec{a}$?

Solution:
If we reflect at the xy-plane, then the vector $\vec{a} = [a_1, a_2, a_3]$ gets changed to $[a_1, a_2, -a_3]$. You can see this by watching the reflection from above. Notice that the first two components stay the same. Do the same process with the other planes. If we next reflect $[a_1, a_2, -a_3]$ off the xz-plane, we get $[a_1, -a_2, -a_3]$. Finally, reflecting this vector off the yz-plane, we get the vector $[-a_1, -a_2, -a_3]$ as desired.

3 a) In order to see whether two data points $\vec{v} = [1, 1, -2]$ and $\vec{w} = [1, -2, 1]$ are correlated, we compute the cosine of the angle between the two vectors. Do this for the vectors $\vec{v}$ and $\vec{w}$.
b) Find two vectors $\vec{a}$ and $\vec{b}$ for which all coordinates are positive such that the angle between them is $\pi/4 = 45°$. In statistics the dot product between $\vec{v}$ and $\vec{w}$ is also called the **covariance** and the lengths $|\vec{v}|$ and $|\vec{w}|$ are called the **standard deviations** of $\vec{v}$ and $\vec{w}$. A data scientist calls the cosine of the angle the **correlation**.

Solution:
a) the cosine of the angle is $\vec{v} \cdot w/(|\vec{v}||\vec{w}|) = -3/6 = -1/2$ so that the angle is $2\pi/3$.
b) An example is $\vec{v} = [1, 1, 1]$ and $\vec{w} = [1, 1, 4 - 3\sqrt{2}]$.

4 a) Find the angle between a space diagonal of a cube and the diagonal in one of its faces.
b) The **hypercube** is also called the **tesseract**. It has vertices $(\pm 1, \pm 1, \pm 1, \pm 1)$. Find the angle between the hyper diagonal connecting $(1, 1, 1, 1)$ with $(-1, -1, -1, -1)$ and the space diagonal

connecting $(1,1,1,1)$ with $(-1,-1,-1,1)$.

Solution:
(a) Consider the 'standard' cube with vertices whose coordinates are either 0 or 1. The line segment from $(0,0,0)$ to $(0,1,1)$ is a diagonal of one of its faces (the face on the yz-plane), while the line segment from $(0,0,0)$ to $(1,1,1)$ is a diagonal of the cube. These line segments can be expressed as the vectors $[0,1,1]$ and $[1,1,1]$, respectively. The cosine of the angle between them is then

$$\cos\theta = \frac{[0,1,1]\cdot[1,1,1]}{||[0,1,1]||\cdot||[1,1,1]||} = \frac{2}{\sqrt{2}\cdot\sqrt{3}} = \frac{2}{\sqrt{6}}.$$

Hence $\theta = \arccos(\frac{2}{\sqrt{6}})$.
(b) The vector connecting $(1,1,1,1)$ to $(-1,-1,-1,-1)$ is $[2,2,2,2]$. The vector connecting $(1,1,1,1)$ with $(-1,-1,-1,1)$ is $[2,2,2,0]$. Thus, the cosine of the angle between the hyper diagonal and the space diagonal is given by

$$\cos\theta = \frac{[2,2,2,2]\cdot[2,2,2,0]}{|[2,2,2,2]||[2,2,2,0]|} = \frac{12}{\sqrt{16}\cdot\sqrt{12}} = \frac{\sqrt{3}}{2}.$$

Finally, we calculate $\theta = \arccos(\frac{\sqrt{3}}{2})$.

5 a) Verify that if $\vec{a},\vec{b}$ are nonzero vectors, then $\vec{c} = |\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ bisects the angle between $\vec{a},\vec{b}$ if $\vec{c}$ is not zero.
b) Verify the parallelogram law $|\vec{a}+\vec{b}|^2+|\vec{a}-\vec{b}|^2 = 2|\vec{a}|^2+2|\vec{b}|^2$.

Solution:

(a) The cosine of the angle between a and c is

$$\begin{aligned}\frac{\vec{a}\cdot\vec{c}}{|\vec{a}|\cdot|\vec{c}|} &= \frac{\vec{a}\cdot(|\vec{a}|\vec{b}+|\vec{b}|\vec{a})}{|\vec{a}|\cdot|\vec{c}|}\\ &= \frac{|\vec{a}|(\vec{a}\cdot\vec{b})+|\vec{a}|^2|\vec{b}|)}{|\vec{a}|\cdot|\vec{c}|} = \frac{(\vec{a}\cdot\vec{b})+|\vec{a}||\vec{b}|}{|\vec{c}|}.\end{aligned}$$

A similar calculation yields that the cosine of the angle between b and c is $\frac{(\vec{b}\cdot\vec{a}+|\vec{b}||\vec{a}|}{|\vec{c}|}$. Because these two expressions are exactly the same, this tells us that either c bisects the angle between a and b or that a and b are colinear. However, if a and b are collinear, c is also collinear with a and b and thus it still bisects the angle between them.

(b) We compute:

$$\begin{aligned}&|\vec{a}+\vec{b}|^2+|\vec{a}-\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})+(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})\\ =\ &(\vec{a}\cdot a+2\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b})+(\vec{a}\cdot\vec{a}-2\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b})\\ =\ &2(\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{b})\\ =\ &2(|\vec{a}|^2+|\vec{b}|^2)\\ =\ &2|\vec{a}|^2+2|\vec{b}|^2.\end{aligned}$$

Main definitions

Two points $P = (a, b, c)$ and $Q = (x, y, z)$ define a **vector** $\vec{v} = [x - a, y - b, z - c]$. We also write $\vec{v} = \vec{PQ}$. The numbers v_1, v_2, v_3 in $\vec{v} = [v_1, v_2, v_3]$ are the **components** of $\vec{v}$. The **length** $|\vec{v}|$ of a vector $\vec{v} = \vec{PQ}$ is defined as the distance $d(P, Q)$ from P to Q. A vector of length 1 is called a **unit vector**. The **addition** is $\vec{u} + \vec{v} = [u_1, u_2, u_3] + [v_1, v_2, v_3] = [u_1+v_1, u_2+v_2, u_3+v_3]$. The **scalar multiple** $\lambda\vec{u} = \lambda[u_1, u_2, u_3] = [\lambda u_1, \lambda u_2, \lambda u_3]$. The difference $\vec{u} - \vec{v}$ can be seen as $\vec{u} + (-\vec{v})$.

The **dot product** of two vectors $\vec{v} = [a, b, c]$ and $\vec{w} = [p, q, r]$ is defined as $\vec{v} \cdot \vec{w} = ap + bq + cr$. The **Cauchy-Schwarz inequality** tells $|\vec{v} \cdot \vec{w}| \leq |\vec{v}||\vec{w}|$.

The **angle** between two nonzero vectors is defined as the unique $\alpha \in [0, \pi]$ satisfying $\vec{v} \cdot \vec{w} = |\vec{v}| \; |\vec{w}| \cos(\alpha)$. Two vectors are called **orthogonal** or **perpendicular** if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. For example, $\vec{v} = [2, 3]$ is orthogonal to $\vec{w} = [-3, 2]$. The vector $\mathrm{P}(\vec{v}) = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$. The **scalar projection** $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}$ is plus or minus the length of the projection of $\vec{v}$ onto $\vec{w}$. The vector $\vec{b} = \vec{v} - P(\vec{v})$ is a vector orthogonal to $\vec{w}$. **Pythagoras tells:** if $\vec{v}$ and $\vec{w}$ are orthogonal, then $|v - w|^2 = |v|^2 + |w|^2$.