Homework 1: Geometry and Distance

This homework is due on Wednesday, 9/11 at the beginning of class.

1 a) Find its center and radius of the sphere S given by $x^2 - 4x + y^2 + 2y + z^2 - 8z = 30$.
b) Find the distance from the center of S defined in a) to the sphere $T : x^2 + y^2 + z^2 = 900$.
c) Find the minimal distance between the spheres S and T. This is the minimal distance between two points x, y where x is in S and y is in T.

Solution:
a) Complete the square:

$$(x-2)^2 + (y+1)^2 + (z-4)^2 = 9$$

Thus, the center of the sphere is at $(2, -1, 4)$ and the radius is $\sqrt{51}$.
b) The sphere $x^2 + y^2 + z^2 = 900$ has radius 30. It contains the center $(2, -1, 4)$ of the sphere S. The distance between the centers is $\sqrt{2^2 + 1^2 + 4^2} = \sqrt{21}$. The center of the sphere S has distance $\sqrt{21}$ from the center of the other sphere and $30 - \sqrt{21}$ is the distance between the center of S and the sphere of radius 30.
c) To get the distance between the two spheres, draw the line connecting the two centers and subtract the correct distances. We get $d = 30 - \sqrt{21} - \sqrt{51}$.

2 a) Find the distance from $P = (-21, -7, -6)$ to $y = 0$.
b) Find the distance from P to the x-axes.
c) Find a point which has distance 5 from the x axes and distance 2 from the yz-plane.

Solution:
a) The distance to the xz-plane is 7. b) The distance to the z axes is $\sqrt{6^2 + 7^2} = 7\sqrt{85}$. c) An example is $(2, 3, 4)$.

3 a) Find an equation of the largest sphere with center $(4, 11, 9)$ that is contained in the first octant $\{x \geq 0, y \geq 0, z \geq 0\}$.
b) Find the equation for the sphere centered at $(6, 10, 8)$ which passes through the center $(4, 11, 9)$ of the sphere in a).

Solution:
a) The closest coordinate plane is the yz plane. It has distance 4, so the radius of the largest sphere must be 4. The equation is $(x-4)^2 + (y-11)^2 + (z-9)^2 = 4^2$.
b) The distance between $(6, 10, 8)$ and $(4, 11, 9)$ is $\sqrt{6}$. Thus, the sphere's equation is $(x-6)^2 + (y-10)^2 + (z-8)^2 = 6$.

4 a) Describe the surface S given by $(x - 2y + z)^2 = 4$ in $\mathbb{R}^3$.

If you like to see a bit of a story behind this, on the website, under "data", you find something about prime numbers related to this surface S.

b) What surface is $x^2 + y^2 - 3 = 0$ in $\mathbb{R}^3$?
c) What is the set $x^2 + y^2 - 3 = 0$ in $\mathbb{R}^2$?

Solution:

a) This is a union of two planes. $x-2y+z=2$ and $x-2y+z=-2$. We will talk about planes later, but you can see that if you fix one of the variables like $z=0$ that you get lines. The set $x-2y+z=2$ therefore has the property that however you cut it with a plane like $x=x_0$ or $y=y_0$ or $z=z_0$, then you end up with a line. This must be a plane.

b) This is a cylinder of radius $\sqrt{3}$ which has the z-axes as center.

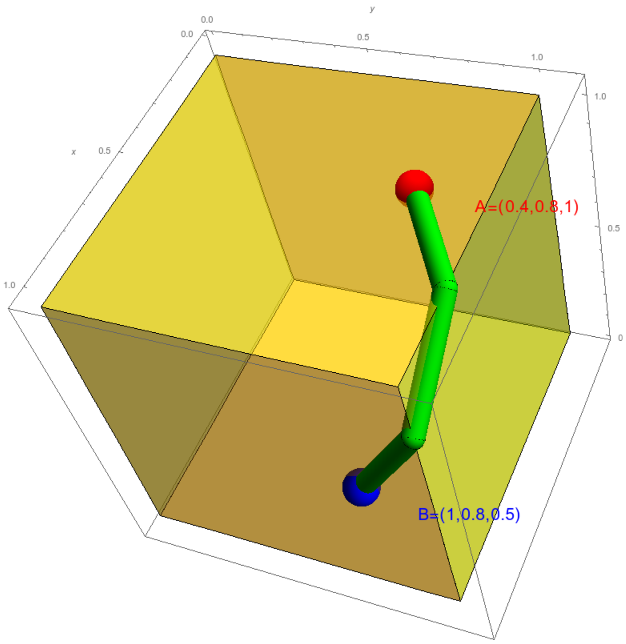
c) This is a circle of radius $\sqrt{3}$ centered at the point $(0,0)$.

5 An ant moves on the unit cube bound by the walls $x=0, x=1, y=0, y=1, z=0, z=1$ from the point $A=(0.4,0.8,1)$ to the point $B=(1,0.8,0.5)$. Compute the length of the two obvious paths, where one passes over three faces, the other only over two. Which one is shorter? See the figures on the third page.

Solution:

Let's look at the first situation, where we move along three faces and where we draw things in the two dimensional plane. The point B has coordinates $(0.8,0.5)$ and the point A has coordinates $(1.6,1.2)$. The distance between them is $\sqrt{0.8^2+0.7^2}=1.06$.

In the second case, we have the points $(0.8,0.5)$ and $(0.8,1.6)$. The distance is 1.1. The first distance is a tiny bit shorter.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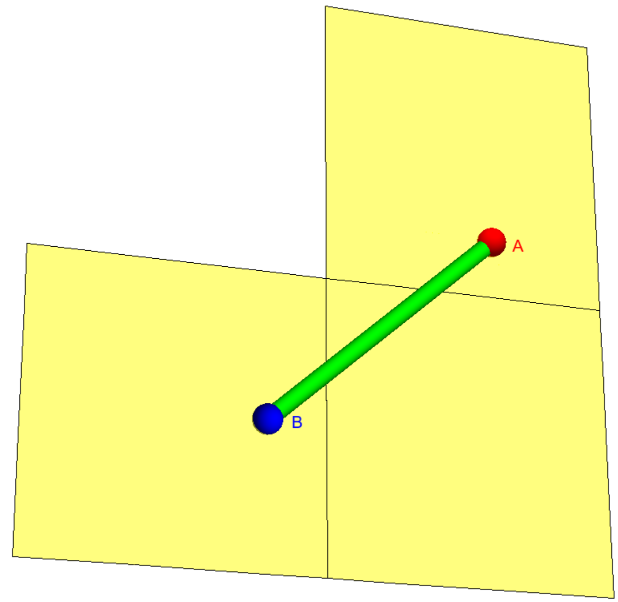
A
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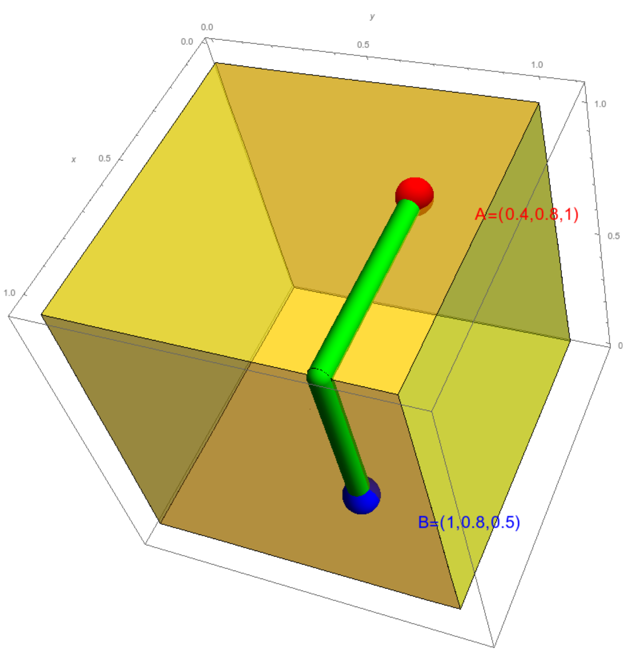
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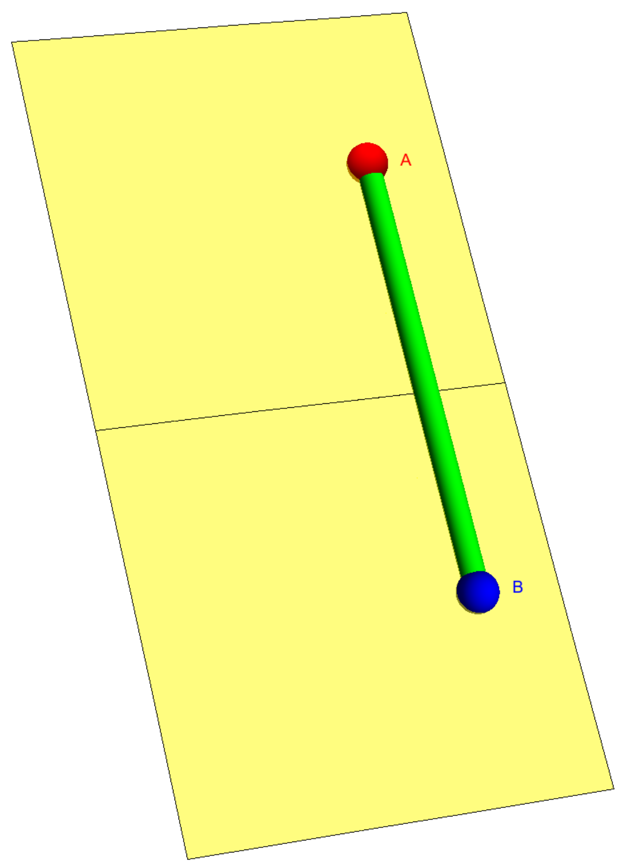
A
B

Main definitions

Points in the **plane** $\mathbb{R}^2$ or **space** $\mathbb{R}^3$ are described using **coordinates** $P = (x, y)$ or $P = (x, y, z)$. Their signs define **quadrants** in the plane or **octants** in space, regions which intersect at the **origin** $O = (0, 0)$ or $O = (0, 0, 0)$ and are separated by **coordinate planes** $\{x = 0\}, \{y = 0\}, \{z = 0\}$ intersecting in **coordinate axes** like the z-axes $\{y = 0, x = 0\}$.

The **Euclidean distance** between two points $P = (x, y, z)$ and $Q = (a, b, c)$ in space is defined as $d(P, Q) = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$. The distance between a point P and a geometric object S is the minimal distance $d(P, Q)$ with Q located on S.

A **circle** of radius r centered at $P = (a, b)$ is the set of points in the plane which have distance r from P. A **sphere** of radius ρ centered at $P = (a, b, c)$ is the set of points in space which have distance ρ from P. The equation of a sphere is $(x-a)^2 + (y-b)^2 + (z-c)^2 = \rho^2$.

To **complete the square** of $x^2+bx+c = 0$, add $(b/2)^2-c$ on both sides to get $(x + b/2)^2 = (b/2)^2 - c$. Solving for x gives $x = -b/2 \pm \sqrt{(b/2)^2 - c}$. **Example:** $x^2 + 8x + y^2 = 9$. **Solution:** Add 16 on both sides to get $x^2+8x+16+y^2 = 25$ which is $(x + 4)^2 + y^2 = 25$, a circle of radius $r = 5$ centered at $(-4, 0)$.