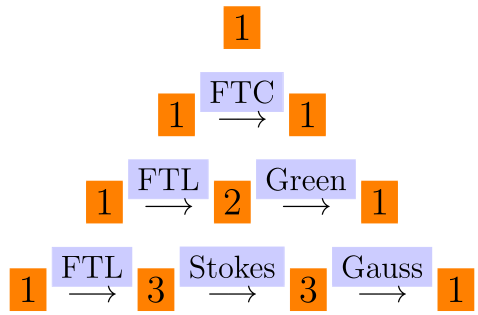
Lecture 32: Overview

All integral theorems are incarnations of **the fundamental theorem of multivariable Calculus**

$$\int_G dF = \int_{\delta G} F$$

where dF is a **derivative** of F and δG is the **boundary** of G.



Fundamental theorem of line integrals: If C is a curve with boundary $\{A, B\}$ and f is a function, then

$$\int_C \nabla f \cdot \vec{dr} = f(B) - f(A) \; .$$

Remarks.
1) For closed curves, the line integral $\int_C \nabla f \cdot \vec{dr}$ is zero.
2) Gradient fields are **path independent**: if $\vec{F} = \nabla f$, then the line integral between two points P and Q does not depend on the path connecting the two points.
3) The theorem justifies the name **conservative** for gradient vector fields.
4) The term "potential" was coined by George Green who lived from 1793-1841.

1 Let $f(x, y, z) = x^2 + y^4 + z$. Find the line integral of the vector field $\vec{F}(x, y, z) = \nabla f(x, y, z)$ along the path $\vec{r}(t) = [\cos(5t), \sin(2t), t^2]$ from $t = 0$ to $t = 2\pi$.
Solution. $\vec{r}(0) = [1, 0, 0]$ and $\vec{r}(2\pi) = [1, 0, 4\pi^2]$ and $f(\vec{r}(0)) = 1$ and $f(\vec{r}(2\pi)) = 1 + 4\pi^2$. The fundamental theorem of line integral gives $\int_C \nabla f \; \vec{dr} = f(r(2\pi)) - f(r(0)) = 4\pi^2$.

Green's theorem. If R is a region with boundary C and $\vec{F}$ is a vector field, then

$$\int\int_R \text{curl}(\vec{F}) \; dxdy = \int_C \vec{F} \cdot \vec{dr} \; .$$

Remarks.
1) The curve is oriented in such a way that the region is to the left.
2) The boundary of the curve can consist of piecewise smooth pieces.
3) If $C : t \mapsto \vec{r}(t) = [x(t), y(t)]$, the line integral is $\int_a^b [P(x(t), y(t)), Q(x(t), y(t))] \cdot [x'(t), y'(t)] \; dt$.
4) Green's theorem was found by George Green (1793-1841) in 1827 and by Mikhail Ostrogradski (1801-1862). It also appeared in a 1846 note of Augustin Cauchy without proof.
5) If $\text{curl}(\vec{F}) = 0$ everywhere in the plane, then the field is a gradient field.
7) $\vec{F}(x, y) = [0, x]$ we get an **area formula**.

2 Find the line integral of the vector field $\vec{F}(x,y) = [x^4 + \sin(x) + y, x + y^3]$ along the path $\vec{r}(t) = [\cos(t), 5\sin(t) + \log(1+\sin(t))]$, where t runs from $t=0$ to $t=\pi$.
Solution. $\mathrm{curl}(\vec{F}) = 0$ implies that the line integral depends only on the end points $(0,1), (0,-1)$ of the path. Take the simpler path $\vec{r}(t) = [-t, 0], -1 \le t \le 1$, which has velocity $\vec{r}\,'(t) = [-1, 0]$. The line integral is $\int_{-1}^{1} [t^4 - \sin(t), -t] \cdot [-1, 0]\; dt = -t^5/5|_{-1}^{1} = -2/5$.

Stokes theorem. If S is a surface with boundary C and $\vec{F}$ is a vector field, then

$$\int\int_S \mathrm{curl}(\vec{F}) \cdot dS = \int_C \vec{F} \cdot d\vec{r} \; .$$

Remarks.
1) Stokes theorem allows to derive Greens theorem: if $\vec{F}$ is z-independent and the surface S is contained in the xy-plane, one obtains the result of Green.
2) The orientation of C is such that if you walk along C and have your head in the direction of the normal vector $\vec{r}_u \times \vec{r}_v$, then the surface to your left.
3) Stokes theorem was found by André Ampère (1775-1836) in 1825 and rediscovered by George Stokes (1819-1903). It appeared first in print in 1854, in an examination question. The theorem had appeared in a letter of Lord Kelvin to Stokes in 1850 already. The first proof was done by Hermann Hankel in 1861.
4) The flux of the curl of $\vec{F}$ only depends on the boundary of S.
5) The flux of the curl through a closed surface like the sphere is zero because the boundary is empty.

3 Compute the line integral of $\vec{F}(x,y,z) = [x^3 + xy, y, z]$ along the polygonal path C connecting the points $(0,0,0), (2,0,0), (2,1,0), (0,1,0)$.
Solution. The path C bounds a surface $S : \vec{r}(u,v) = [u,v,0]$ parameterized by $R = [0,2] \times [0,1]$. By Stokes theorem, the line integral is equal to the flux of $\mathrm{curl}(\vec{F})(x,y,z) = [0,0,-x]$ through S. The normal vector of S is $\vec{r}_u \times \vec{r}_v = [1,0,0] \times [0,1,0] = [0,0,1]$ so that $\int\int_S \mathrm{curl}(\vec{F}) \cdot d\vec{S} = \int_0^2 \int_0^1 [0,0,-u] \cdot [0,0,1]\; dudv = \int_0^2 \int_0^1 -u \; dudv = -2$.

Divergence theorem: If S is the boundary of a region E in space and $\vec{F}$ is a vector field, then

$$\int\int\int_B \mathrm{div}(\vec{F})\; dV = \int\int_S \vec{F} \cdot d\vec{S} \; .$$

Remarks.
1) The divergence theorem is also called **Gauss theorem**.
2) It can be helpful to determine the flux of vector fields through surfaces.
3) It was discovered in 1764 by Joseph Louis Lagrange (1736-1813), later it was rediscovered by Carl Friedrich Gauss (1777-1855) and by George Green.
4) For divergence free vector fields $\vec{F}$, the flux through a closed surface is zero. Such fields $\vec{F}$ are also called **incompressible** or **source free**.

4 Compute the flux of the vector field $\vec{F}(x,y,z) = [-x, y, z^2]$ through the boundary S of the rectangular box $[0,3] \times [-1,2] \times [1,2]$.
Solution. By Gauss theorem, the flux is equal to the triple integral of $\mathrm{div}(\vec{F}) = 2z$ over the box: $\int_0^3 \int_{-1}^2 \int_1^2 2z \; dxdydz = (3-0)(2-(-1))(4-1) = 27$.