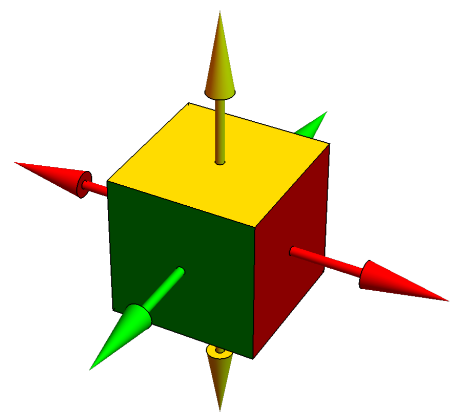
Lecture 31: Divergence theorem

The divergence theorem completes the list of integral theorems in space. It deals with a solid E in space with boundary surface S oriented outwards and a vector field $\vec{F}$.

Divergence Theorem.

$$\iiint_E \text{div}(\vec{F})\ dV = \iint_S \vec{F}\ \cdot d\vec{S}\ .$$

To see why this is true look at a small box $[x, x+dx] \times [y, y+dy] \times [z, z+dz]$. The flux of $\vec{F} = [P, Q, R]$ through the faces perpendicular to the x-axes is $[\vec{F}(x+dx, y, z) \cdot [1, 0, 0] + \vec{F}(x, y, z) \cdot [-1, 0, 0]]dydz = P(x+dx, y, z) - P(x, y, z) = P_x\ dxdydz$. Similarly, the flux through the y-boundaries is $P_y\ dydxdz$ and the flux through the two z-boundaries is $P_z\ dzdxdy$. The total flux through the faces of the cube is $(P_x + P_y + P_z)\ dxdydz = \text{div}(\vec{F})\ dxdydz$.



The theorem explains what divergence means. The average divergence over a small cube is equal the flux of the field through the boundary of the cube. If this is positive, then more field leaves cube than enters the cube. There is field "generated" inside. The divergence measures the expansion of the field.

1 Let $\vec{F}(x, y, z) = [x, y, z]$ and let S be sphere. The divergence of $\vec{F}$ is the constant function $\text{div}(\vec{F}) = 3$ and $\int\int\int_G \text{div}(\vec{F})\ dV = 3 \cdot 4\pi/3 = 4\pi$. The flux through the boundary is $\int\int_S \vec{r} \cdot \vec{r}_u \times \vec{r}_v\ dudv = \int\int_S |\vec{r}(u, v)|^2 \sin(v)\ dudv = \int_0^{\pi} \int_0^{2\pi} \sin(v)\ dudv = 4\pi$ also. We see that the divergence theorem allows us to compute the area of the sphere from the volume of the enclosed ball or compute the volume from the surface area.

2 What is the flux of the vector field $\vec{F}(x, y, z) = [2x, 3z^2 + y, \sin(x)]$ through the solid $G = [0, 3] \times [0, 3] \times [0, 3] \setminus ([0, 3] \times [1, 2] \times [1, 2] \cup [1, 2] \times [0, 3] \times [1, 2] \cup [0, 3] \times [0, 3] \times [1, 2])$ which is a cube where three perpendicular cubic holes have been removed? **Solution:** Use the

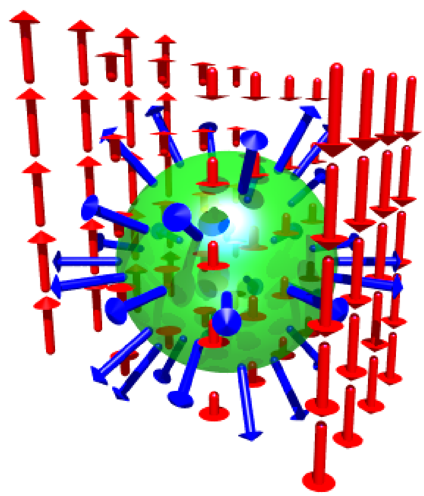
divergence theorem: $\text{div}(\vec{F}) = 2$ and so $\int\int\int_G \text{div}(\vec{F})\ dV = 2\int\int\int_G\ dV = 2\text{Vol}(G) = 2(27-7) = 40$. Note that the flux integral here would be over a complicated surface over dozens of rectangular planar regions.

3 Find the flux of curl(F) through a torus if $\vec{F} = [yz^2, z + \sin(x) + y, \cos(x)]$ and the torus has the parametrization

$$\vec{r}(\theta, \phi) = [(2 + \cos(\phi))\cos(\theta), (2 + \cos(\phi))\sin(\theta), \sin(\phi)] .$$

Solution: The answer is 0 because the divergence of curl(F) is zero. By the divergence theorem, the flux is zero. In general, the flux of the curl of a field through a closed surface is zero.

4 Similarly as Green's theorem allowed to calculate the area of a region by integration along the boundary, the volume of a region can be computed as a flux integral: take the vector field $\vec{F}(x, y, z) = [x, 0, 0]$ which has divergence 1. The flux of this vector field through the boundary of a solid region is equal to the volume of the solid: $\int\int_{\delta G}[x, 0, 0] \cdot d\vec{S} = \text{Vol}(G)$.



5 How heavy are we at distance r from the center of the earth? **Solution:** The law of gravity can be formulated as $\text{div}(\vec{F}) = 4\pi\rho$, where ρ is the mass density. We assume that the earth is a ball of radius R. By rotational symmetry, the gravitational force is normal to the surface: $\vec{F}(\vec{x}) = \vec{F}(r)\vec{x}/||\vec{x}||$. The flux of $\vec{F}$ through a ball of radius r is $\int\int_{S_r}\vec{F}(x) \cdot d\vec{S} = 4\pi r^2 \vec{F}(r)$. By the **divergence theorem**, this is $4\pi M_r = 4\pi\int\int\int_{B_r}\rho(x)\ dV$, where M_r is the mass of the material inside S_r. We have $(4\pi)^2\rho r^3/3 = 4\pi r^2\vec{F}(r)$ for $r < R$ and $(4\pi)^2\rho R^3/3 = 4\pi r^2\vec{F}(r)$ for $r \geq R$. Inside the earth, the gravitational force $\vec{F}(r) = 4\pi\rho r/3$. Outside the earth, it satisfies $\vec{F}(r) = M/r^2$ with $M = 4\pi R^3\rho/3$.