Lecture 30: Stokes theorem

The **boundary** of a surface is a curve oriented so that the surface is to the "left" if the normal vector to the surface is pointing "up". In other words, the velocity vector v, a vector w pointing towards the surface and the normal vector n to the surface form a right handed coordinate system.

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Stokes theorem: Let S be a surface bounded by a curve C and $\vec{F}$ be a vector field. Then

$$\int\int_S \text{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} .$$

1 Let $\vec{F}(x,y,z) = [-y,x,0]$ and let S be the upper semi hemisphere, then $\text{curl}(\vec{F})(x,y,z) = [0,0,2]$. The surface is parameterized by $\vec{r}(u,v) = [\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)]$ on $G = [0,2\pi] \times [0,\pi/2]$ and $\vec{r}_u \times \vec{r}_v = \sin(v)\vec{r}(u,v)$ so that $\text{curl}(\vec{F})(x,y,z) \cdot \vec{r}_u \times \vec{r}_v = \cos(v)\sin(v)2$. The integral $\int_0^{2\pi}\int_0^{\pi/2} \sin(2v)\, dvdu = 2\pi$.
The boundary C of S is parameterized by $\vec{r}(t) = [\cos(t), \sin(t), 0]$ so that $d\vec{r} = \vec{r}\,'(t)dt = [-\sin(t), \cos(t), 0]\, dt$ and $\vec{F}(\vec{r}(t))\, \vec{r}\,'(t)dt = \sin(t)^2 + \cos^2(t) = 1$. The line integral $\int_C \vec{F} \cdot d\vec{r}$ along the boundary is 2π.

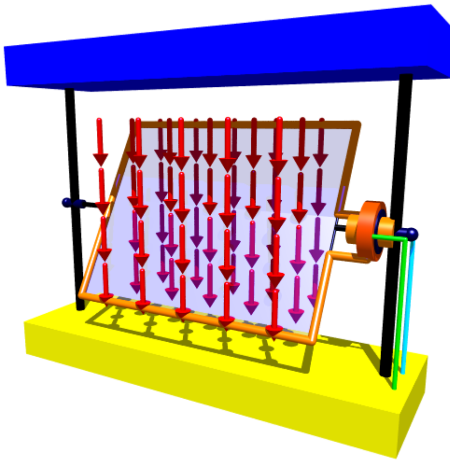
2 If S is a surface in the xy-plane and $\vec{F} = [P,Q,0]$ has zero z component, then $\text{curl}(\vec{F}) = [0,0,Q_x - P_y]$ and $\text{curl}(\vec{F}) \cdot d\vec{S} = Q_x - P_y\; dxdy$. In this case, Stokes theorem can be seen as a consequence of Green's theorem. The vector field F induces a vector field on the surface such that its $2D$ curl is the normal component of $\text{curl}(F)$. The reason is that the third component $Q_x - P_y$ of $\text{curl}(\vec{F})[R_y - Q_z, P_z - R_x, Q_x - P_y]$ is the two dimensional curl: $\vec{F}(\vec{r}(u,v)) \cdot [0,0,1] = Q_x - P_y$. If C is the boundary of the surface, then $\int\int_S \vec{F}(\vec{r}(u,v)) \cdot [0,0,1]\, dudv = \int_C \vec{F}(\vec{r}(t))\, \vec{r}\,'(t)dt$.

3 Calculate the flux of the curl of $\vec{F}(x,y,z) = [-y,x,0]$ through the surface parameterized by $\vec{r}(u,v) = [\cos(u)\cos(v), \sin(u)\cos(v), \cos^2(v) + \cos(v)\sin^2(u+\pi/2)]$. Because the surface has the same boundary as the upper half sphere, the integral is again 2π as in the above example.

A surface is closed if it bounds a solid.

The flux of the curl of a vector field through a closed surface is zero.

4 The electric field E and the magnetic field B are linked by the **Maxwell equation** $\text{curl}(\vec{E}) = -\frac{1}{c}\dot{B}$. Take a closed wire C which bounds a surface S and consider $\int\int_S B \cdot dS$, the flux of the magnetic field through S. The flux change is related with a voltage using Stokes theorem: $d/dt \int\int_S B \cdot dS = \int\int_S \dot{B} \cdot dS = \int\int_S -c \ \text{curl}(\vec{E}) \cdot d\vec{S} = -c \int_C \vec{E} \ d\vec{r} = U$, where U is the voltage measured at the cut-up wire. The flux can be changed by turn around a magnet around the wire or the wire inside the magnet, we get an electric voltage. Stokes theorem explains why we can generate electricity from motion.



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5 Find $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x,y,z) = [x^2y, x^3/3, xy]$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above. **Solution.** The curl of F is $\text{curl}(F) = (x, -y, 0)$. We can parametrize the hyperbolic paraboloid as $\vec{r}(u,v) = (u\cos(v), u\sin(v), -u^2\cos(2v))$. $\vec{r}_u \times \vec{r}_v = [2u^2\cos(v), -2u^2\sin(v), u]$. $\vec{F}(\vec{r}(u,v)) = [u\cos(v), -u\sin(v), 0]$. $\vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v) = 2u^3$. The integral is $\int_0^1 \int_0^{2\pi} -2r^3 \ d\theta dr = \pi$.

6 Evaluate the flux of $\vec{F}(x,y,z) = [xe^{y^2}z^3 + 2xyze^{x^2+z}, x + z^2e^{x^2+z}, ye^{x^2+z} + ze^x]$ through the part S of the ellipsoid $x^2 + y^2/4 + (z+1)^2 = 2$, $z > 0$ oriented so that the normal vector points upwards. **Solution.** Stokes theorem assures that the flux integral is equal to the line integral along the boundary of the surface. The boundary is the ellipse $\vec{r}(t) = [\cos(t), 2\sin(t)]$, $0 \leq t \leq 2\pi$. The vector field on the xy-plane $z = 0$ is

$$\vec{F}(x,y,0) = [0, x, ye^{x^2}] \ .$$

To compute the line integral of this vector field along the boundary curve, compute $\vec{r}'(t) = [-\sin(t), 2\cos(t), 0]$ and $\vec{F}(\vec{r}(t)) = [0, \cos(t), 2\sin(t)e^{\sin^2(t)}]$. The dot product is the power $2\cos^2(t)$. Integrating this over $[0, 2\pi]$ gives 2π.