Lecture 29: Curl, Divergence and Flux

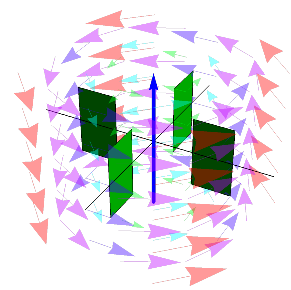
The **curl** of $\vec{F} = [P, Q]$ is $Q_x - P_y$, a scalar field. The **curl** of $\vec{F} = [P, Q, R]$ is

$$\mathrm{curl}(P, Q, R) = [R_y - Q_z, P_z - R_x, Q_x - P_y] .$$

We can write $\mathrm{curl}(\vec{F}) = \nabla \times \vec{F}$. Fields of zero curl are called **irrotational**.

1 The curl of the vector field $[x^2 + y^5, z^2, x^2 + z^2]$ is $[-2z, -2x, -5y^4]$.

If you place a "paddle wheel" pointing into the direction v, its rotation speed $\vec{F} \cdot \vec{v}$. The direction in which the wheel turns fastest, is the direction of $\mathrm{curl}(\vec{F})$. The angular velocity is the magnitude of the curl.



The **divergence** of $\vec{F} = [P, Q, R]$ is $\mathrm{div}([P, Q, R]) = \nabla \cdot \vec{F} = P_x + Q_y + R_z$. The **divergence** of $\vec{F} = [P, Q]$ is $\mathrm{div}(P, Q) = \nabla \cdot \vec{F} = P_x + Q_y$.

The divergence measures the "expansion" of a field. Fields of zero divergence are **incompressible**. With $\nabla = [\partial_x, \partial_y, \partial_z]$, we can write $\mathrm{curl}(\vec{F}) = \nabla \times \vec{F}$ and $\mathrm{div}(\vec{F}) = \nabla \cdot \vec{F}$.

$$\Delta f = \mathrm{div}(\mathrm{grad}(f)) = f_{xx} + f_{yy} + f_{zz} .$$

is the Laplacian of f. One also writes $\Delta f = \nabla^2 f$ because $\nabla \cdot (\nabla f) = \mathrm{div}(\mathrm{grad}(f))$.

From $\nabla \cdot \nabla \times \vec{F} = 0$ and $\nabla \times \nabla \vec{F} = \vec{0}$, we get

$\mathrm{div}(\mathrm{curl}(\vec{F})) = 0$, $\mathrm{curl}(\mathrm{grad}(f)) = \vec{0}$.

2 **Question:** Is there a vector field $\vec{G}$ such that $\vec{F} = [x + y, z, y^2] = \mathrm{curl}(\vec{G})$?
Answer: No, because $\mathrm{div}(\vec{F}) = 1$ is incompatible with $\mathrm{div}(\mathrm{curl}(\vec{G})) = 0$.

3 Show that in simply connected region, every irrotational and incompressible field can be written as a vector field $\vec{F} = \mathrm{grad}(f)$ with $\Delta f = 0$. Proof. Since $\vec{F}$ is irrotational, there exists a function f satisfying $F = \mathrm{grad}(f)$. Now, $\mathrm{div}(F) = 0$ implies $\mathrm{divgrad}(f) = \Delta f = 0$.

4 Find an example of a field which is both incompressible and irrotational. Solution. Find f which satisfies the Laplace equation $\Delta f = 0$, like $f(x,y) = x^3 - 3xy^2$, then look at its gradient field $\vec{F} = \nabla f$. In that case, this gives

$$\vec{F}(x,y) = [3x^2 - 3y^2, -6xy] \; .$$

We have now all the derivatives together. In dimension d, there are d fundamental derivatives.

$$1$$

$$1 \xrightarrow{\text{grad}} 1$$

$$1 \xrightarrow{\text{grad}} 2 \xrightarrow{\text{curl}} 1$$

$$1 \xrightarrow{\text{grad}} 3 \xrightarrow{\text{curl}} 3 \xrightarrow{\text{div}} 1$$

If a surface S is parametrized as $\vec{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$ over a domain R in the uv-plane and $\vec{F}$ is a vector field, then the **flux integral** of $\vec{F}$ through S is

$$\int \int_F \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \; dudv \; .$$

1 Compute the flux of $\vec{F}(x,y,z) = [0,1,z^2]$ through the upper half sphere S parametrized by

$$\vec{r}(u,v) = [\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)] \; .$$

Solution. We have $\vec{r}_u \times \vec{r}_v = -\sin(v)\vec{r}$ and $\vec{F}(\vec{r}(u,v)) = [0,1,\cos^2(v)]$ so that

$$\int_0^{2\pi} \int_0^{\pi} -[0,1,\cos^2(v)] \cdot [\cos(u)\sin^2(v), \sin(u)\sin^2(v), \cos(v)\sin(v)] \; dudv \; .$$

The flux integral is $\int_0^{2\pi} \int_{\pi/2}^{\pi} -\sin^2(v)\sin(u) - \cos^3(v)\sin(v) \; dudv$ which is $-\int_{\pi/2}^{\pi} \cos^3 v \sin(v) \; dv = \cos^4(v)/4|_0^{\pi/2} = -1/4$.

2 Calculate the flux of $\vec{F}(x,y,z) = [1,2,4z]$ through the paraboloid $z = x^2 + y^2$ lying above the region $x^2 + y^2 \leq 1$. **Solution:** We can parametrize the surface as $\vec{r}(r,\theta) = [r\cos(\theta), r\sin(\theta), r^2]$ where $\vec{r}_r \times \vec{r}_\theta = [-2r^2\cos(\theta), -2r^2\sin(\theta), r]$ and $\vec{F}(\vec{r}(u,v)) = [1,2,4r^2]$. We get $\int_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^1 (-2r^2\cos(v) - 4r^2\sin(v) + 4r^3) \; drd\theta = 2\pi$.

3 Evaluate the flux integral $\int\int_S \text{curl}(F) \cdot d\vec{S}$ for $\vec{F}(x,y,z) = [xy, yz, zx]$, where S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $[0,1] \times [0,1]$ and has an upward orientation. **Solution:** $\text{curl}(F) = [-y,-z,-x]$. The parametrization $\vec{r}(u,v) = [u, v, 4 - u^2 - v^2]$ gives $r_u \times r_v = [2u, 2v, 1]$ and $\text{curl}(F)(\vec{r}(u,v)) = [-v, u^2 + v^2 - 4, -u]$. The flux integral is $\int_0^1 \int_0^1 [-2uv + 2v(u^2+v^2-4) - u] \; dvdu = -1/2 + 1/3 + 1/2 - 4 - 1/2 = -25/6$.