Lecture 28: Green's theorem

The **curl** of a vector field $\vec{F}(x,y) = [P(x,y), Q(x,y)]$ is defined as the scalar field

$$\text{curl}(F)(x,y) = Q_x(x,y) - P_y(x,y) .$$

The function curl(F) measures the **vorticity** of the vector field. One can write $\nabla \times \vec{F} = \text{curl}(\vec{F})$ because the two dimensional cross product of (∂_x, ∂_y) with $\vec{F} = [P,Q]$ is the scalar $Q_x - P_y$.

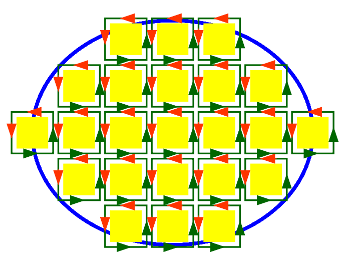
1 For $\vec{F}(x,y) = [-y, x]$ we have curl$(F)(x,y) = 2$.

2 If $\vec{F}(x,y) = \nabla f$ is a gradient field then the curl is zero because if $P(x,y) = f_x(x,y), Q(x,y) = f_y(x,y)$ and curl$(F) = Q_x - P_y = f_{yx} - f_{xy} = 0$ by Clairaut's theorem.

Green's theorem: If $\vec{F}(x,y) = [P(x,y), Q(x,y)]$ is a vector field and R is a region for which the boundary C is parametrized so that R is "to the left", then

$$\int_C \vec{F} \cdot d\vec{r} = \int\int_G \text{curl}(F) \; dxdy .$$

Proof. The integral of $\vec{F}$ along the boundary of $G = [x, x+\epsilon] \times [y, y+\epsilon]$ is $\int_0^\epsilon P(x+t,y)dt + \int_0^\epsilon Q(x+\epsilon, y+t) \; dt - \int_0^\epsilon P(x+t, y+\epsilon) \; dt - \int_0^\epsilon Q(x, y+t) \; dt$. Because $Q(x+\epsilon, y) - Q(x,y) \sim Q_x(x,y)\epsilon$ and $P(x, y+\epsilon) - P(x,y) \sim P_y(x,y)\epsilon$, this is is $(Q_x - P_y)\epsilon^2 \sim \int_0^\epsilon \int_0^\epsilon \text{curl}(F) \; dxdy$. All identities hold in the limit $\epsilon \to 0$.



A general region G can be cut into small squares of size ϵ. Summing up all the line integrals around the boundaries gives the line integral around the boundary because in the interior, the line integrals cancel. Summing up the vortex strength $Q_x - P_y$ on the squares is a Riemann sum approximation of the double integral. The boundary integrals converge to the line integral of C.

George Green lived from 1793 to 1841. He was a physicist a self-taught mathematician and miller.

3 If $\vec{F}$ is a gradient field then both sides of Green's theorem are zero: $\int_C \vec{F} \cdot d\vec{r}$ is zero by the fundamental theorem for line integrals and $\int\int_G \text{curl}(F) \cdot dA$ is zero because curl$(F) =$ curl(grad$(f)) = 0$.

The already established the Clairaut identity

$\text{curl}(\text{grad}(f)) = 0$

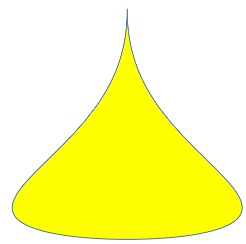
It can also remembered as $\nabla \times \nabla f$ noting that the cross product of two identical vectors is 0. Treating ∇ as a vector is **nabla calculus**.

4 Find the line integral of $\vec{F}(x,y) = [x^2 - y^2, 2xy] = [P,Q]$ along the boundary of the rectangle $[0,2] \times [0,1]$. Solution: $\text{curl}(\vec{F}) = Q_x - P_y = 2y + 2y = 4y$ so that $\int_C \vec{F}\ d\vec{r} = \int_0^2 \int_0^1 4y\ dydx = 2y^2|_0^1 x|_0^2 = 4$.

Find the area of the region enclosed by

5 $$\vec{r}(t) = [\frac{\sin(\pi t)^2}{t}, t^2 - 1]$$

for $-1 \leq t \leq 1$. To do so, use Greens theorem with the vector field $\vec{F} = [0, x]$.



6 An important application of Green is to **compute area**. With the vector fields $\vec{F}(x,y) = [P,Q] = [-y, 0]$ or $\vec{F}(x,y) = [0,x]$ have vorticity $\text{curl}(\vec{F})(x,y) = 1$. For $\vec{F}(x,y) = [0,x]$, the right hand side in Green's theorem is the **area** of G:

$$\text{Area}(G) = \int_C [0, x(t)] \cdot [x'(t), y'(t)]\ dt\ .$$

7 Let G be the region under the graph of a function $f(x)$ on $[a,b]$. The line integral around the boundary of G is 0 from $(a,0)$ to $(b,0)$ because $\vec{F}(x,y) = [0,0]$ there. The line integral is also zero from $(b,0)$ to $(b,f(b))$ and $(a,f(a))$ to $(a,0)$ because $N = 0$. The line integral along the curve $(t, f(t))$ is $-\int_a^b [-y(t), 0] \cdot [1, f'(t)]\ dt = \int_a^b f(t)\ dt$. Green's theorem confirms that this is the area of the region below the graph.

It had been a consequence of the fundamental theorem of line integrals that

If $\vec{F}$ is a gradient field then $\text{curl}(F) = 0$ everywhere.

Is the converse true? Here is the answer:

A region R is called **simply connected** if every closed loop in R can be pulled together to a point in R.

If $\text{curl}(\vec{F}) = 0$ in a simply connected region G, then $\vec{F}$ is a gradient field.

Proof. Given a closed curve C in G enclosing a region R. Green's theorem assures that $\iint_R \text{curl}(\vec{F}))(x,y)\ dx$ $\int_C \vec{F}\ d\vec{r} = 0$. So $\vec{F}$ has the closed loop property in G, line integrals are path independent and $\vec{F}$ is a gradient field.