Lecture 27: Fundamental theorem of line integrals

If $\vec{F}$ is a vector field in the plane or in space and $C : t \mapsto \vec{r}(t)$ is a curve defined on the interval $[a, b]$ then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$$

is called the **line integral** of $\vec{F}$ along the curve C.

The following theorem generalizes the fundamental theorem of calculus to higher dimensions:

Fundamental theorem of line integrals: If $\vec{F} = \nabla f$, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = f(\vec{r}(b)) - f(\vec{r}(a)) \; .$$

The proof of the fundamental theorem uses the chain rule in the second equality and the fundamental theorem of calculus in the third equality of the following identities:

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\; dt = f(\vec{r}(b)) - f(\vec{r}(a)) \; .$$

1 Let $\vec{F}(x, y) = [2xy^2 + 3x^2, 2yx^2]$ be the force field produced by the water near the rheinfalls in Switzerland. Find the line integral along a line from $(0, 0)$ to $(2, 1)$. Solution. We find a potential $f(x, y) = x^2y^2 + x^3$ and instead compute the difference of the potential values which is 12.

2 Let $\vec{F}(x,y) = [2xy^2 + 3x^2, 2yx^2]$. Find a potential f of $\vec{F} = [P,Q]$.
Solution: The potential function $f(x,y)$ satisfies $f_x(x,y) = 2xy^2 + 3x^2$ and $f_y(x,y) = 2yx^2$. Integrating the second equation gives $f(x,y) = x^2y^2 + h(x)$. Partial differentiation with respect to x gives $f_x(x,y) = 2xy^2 + h'(x)$ which should be $2xy^2 + 3x^2$ so that we can take $h(x) = x^3$. The potential function is $f(x,y) = x^2y^2 + x^3$. Find g, h from $f(x,y) = \int_0^x P(t,y)\,dt + h(y)$ and $f_y(x,y) = g(x,y)$.

3 Here is an enigma. Let $\vec{F}(x,y) = [P,Q] = [\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}]$. It is a gradient field because $f(x,y) = \arctan(y/x)$ has the property that $f_x = (-y/x^2)/(1+y^2/x^2) = P$, $f_y = (1/x)/(1+y^2/x^2) = Q$. However, the line integral $\int_\gamma \vec{F}\,\vec{dr}$, where γ is the unit circle is

$$\int_0^{2\pi} [\frac{-\sin(t)}{\cos^2(t)+\sin^2(t)}, \frac{\cos(t)}{\cos^2(t)+\sin^2}] \cdot [-\sin(t), \cos(t)]\,dt$$

which is $\int_0^{2\pi} 1\,dt = 2\pi$. What is wrong?
Solution: note that the potential f as well as the vector-field F are not differentiable everywhere. The curl of F is zero except at $(0,0)$, where it is not defined. The region in which $\vec{F}$ is defined is not simply connected.