Lecture 26: Line integrals

If $\vec{F}$ is a vector field in the plane or in space and $C : t \mapsto \vec{r}(t)$ is a curve defined on the interval $[a, b]$ then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$$

is called the **line integral** of $\vec{F}$ along the curve C.

The short-hand notation $\int_C \vec{F} \cdot d\vec{r}$ is also used. In physics, if $\vec{F}(x, y, z)$ is a force field, then $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$ is called **power** and the line integral $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$ is called **work**. In electrodynamics, if $\vec{F}(x, y, z)$ is an electric field, then the line integral $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$ gives the **electric potential**.

The line integral $\int_C \vec{F} \cdot d\vec{r}$ is independent of the parametrization of the curve but unlike arc length, it depends in which direction we go over the curve.

1 Let $C : t \mapsto \vec{r}(t) = [\cos(t), \sin(t)]$ be a circle parametrized by $t \in [0, 2\pi]$ and let $\vec{F}(x, y) = [-y, x]$. Calculate the line integral $I = \int_C \vec{F}(\vec{r}) \cdot d\vec{r}$.
Solution: We have $I = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = \int_0^{2\pi} [-\sin(t), \cos(t)] \cdot [-\sin(t), \cos(t)]\; dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t)\; dt = 2\pi$

2 Let $\vec{r}(t)$ be a curve given in polar coordinates as $r(t) = \cos(t), \phi(t) = t$ defined on $[0, \pi]$. Let $\vec{F}$ be the vector field $\vec{F}(x, y) = [-xy, 0]$. Calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$. **Solution:** In Cartesian coordinates, the curve is $\vec{r}(t) = [\cos^2(t), \cos(t)\sin(t)]$. The velocity vector is then $\vec{r}\,'(t) = [-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t)) = (x(t), y(t)]$. The line integral is

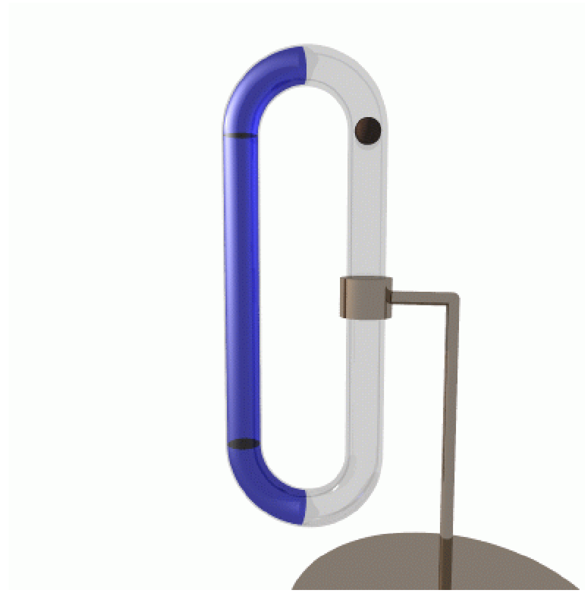
$$\int_0^\pi \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = \int_0^\pi [\cos^3(t)\sin(t), 0] \cdot [-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t)]\; dt$$

$$= \; -2\int_0^\pi \sin^2(t)\cos^4(t)\; dt = -2(t/16 + \sin(2t)/64 - \sin(4t)/64 - \sin(6t)/192)|_0^\pi = -\pi/8 \,.$$

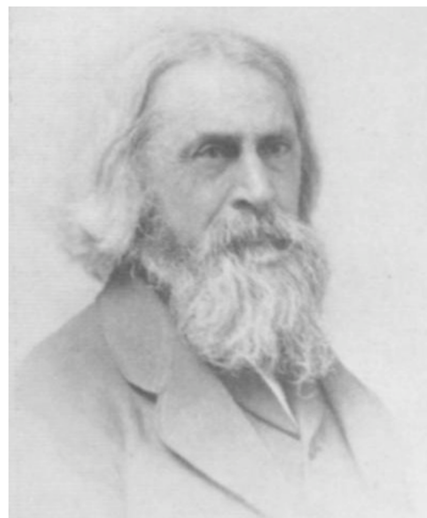
3 Let $f(x, y, z)$ be the temperature distribution in a room and let $\vec{r}(t)$ the path of a fly in the room, then $f(\vec{r}(t))$ is the temperature, the fly experiences at the point $\vec{r}(t)$ at time t. The change of temperature for the fly is $\frac{d}{dt} f(\vec{r}(t))$. The line-integral of the temperature gradient ∇f along the path of the fly coincides with the temperature difference between the end point and initial point.

Something to think about:

A device which implements a non gradient force field is called a **perpetual motion machine**. It realizes a force field for which along some closed loops the energy gain is nonnegative. By possibly changing the direction, the energy change is positive. The first law of thermodynamics forbids the existence of such a machine. It is informative to contemplate some of the ideas people have come up with and to see why they don't work. Here is an example: consider a O-shaped pipe which is filled only on the right side with water. A wooden ball falls on the right hand side in the air and moves up in the water.



Why does this "perpetual motion machine" not work? The former Harvard professor Benjamin Peirce refers in his book "A system of analytic mechanics" of 1855 to the "antropic principle". "Such a series of motions would receive the technical name of a "perpetual motion" by which is to be understood, that of a system which would constantly return to the same position, with an increase of power, unless a portion of the power were drawn off in some way and appropriated, if it were desired, to some species of work. A constitution of the fixed forces, such as that here supposed and in which a perpetual motion would possible, may not, perhaps, be incompatible with the unbounded power of the Creator; but, if it had been introduced into nature, it would have proved destructive to human belief, in the spiritual origin of force, and the necessity of a First Cause superior to matter, and would have subjected the grand plans of Divine benevolence to the will and caprice of man".



Nonconservative fields can also be generated by **optical illusion** as **M.C. Escher** did. The illusion suggests the existence of a force field which is not conservative. Can you figure out how Escher's pictures "work"?