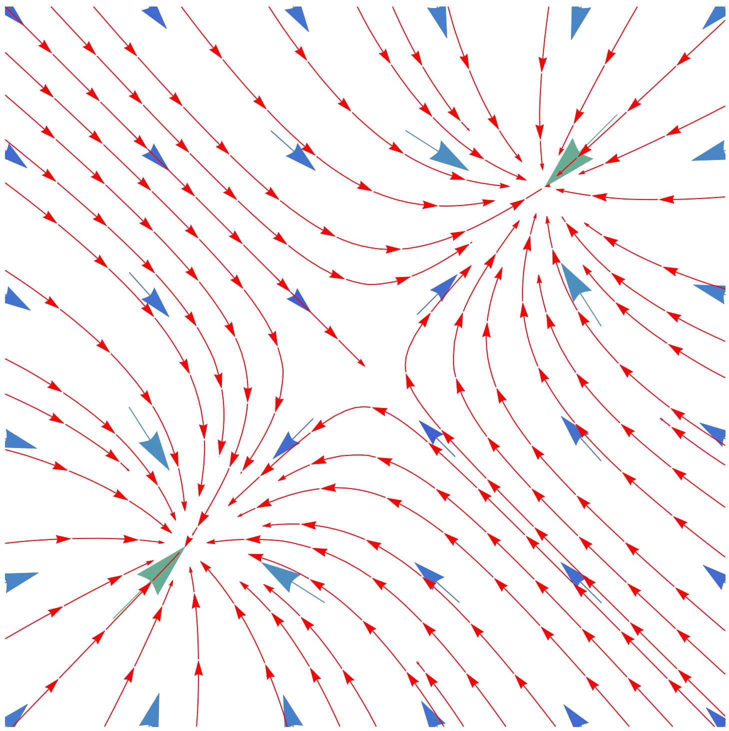
# Lecture 25: Vector fields

A **vector field** in the plane is a map, which assigns to each point $(x, y)$ a vector $\vec{F}(x, y) = [P(x, y), Q(x, y)]$. A vector field in space is a map, which assigns to $(x, y, z)$ in space a vector $\vec{F}(x, y, z) = [P(x, y, z), Q(x, y, z), R(x, y, z)]$.

For example, $\vec{F}(x, y) = [x - 1, y]/((x - 1)^2 + y^2)^{3/2} - [x + 1, y]/((x + 1)^2 + y^2)^{3/2}$ is the electric field of positive and negative point charge. It is called the **dipole field**. It is shown in the picture below:



If $f(x, y)$ is a function of two variables, then $\vec{F}(x, y) = \nabla f(x, y)$ is called a **gradient field**. Gradient fields in space are of the form $\vec{F}(x, y, z) = \nabla f(x, y, z)$.

When is a vector field a gradient field? If $\vec{F}(x, y) = [P(x, y), Q(x, y)] = \nabla f(x, y) = [f_x(x, y), f_y(x, y)]$ then $Q_x(x, y) = P_y(x, y)$ by Clairaut. If this does not hold at some point, then $F$ is no gradient field.

**Clairaut test**: if $Q_x(x, y) - P_y(x, y)$ is not zero at some point, then $\vec{F}(x, y) = [P(x, y), Q(x, y)]$ is not a gradient field.

We will see next week that the condition $\text{curl}(F) = Q_x - P_y = 0$ is also necessary for $\vec{F}$ to be a gradient field. In class, we see more examples on how to construct the potential $f$ from the gradient field $F$.

1 Is the vector field $\vec{F}(x,y) = [P,Q] = [3x^2y + y + 2, x^3 + x - 1]$ a gradient field? **Solution**: the Clairaut test shows $Q_x - P_y = 0$. We integrate the equation $f_x = P = 3x^2y + y + 2$ and get $f(x,y) = 2x + xy + x^3y + c(y)$. Now take the derivative of this with respect to $y$ to get $x + x^2 + c'(y)$ and compare with $x^3 + x - 1$. We see $c'(y) = -1$ and so $c(y) = -y + c$. We see the solution $\boxed{x^3y + xy - y + 2x}$.

2 Is the vector field $\vec{F}(x,y) = [xy, 2xy^2]$ a gradient field? **Solution:** No: $Q_x - P_y = 2y^2 - x$ is not zero.

Vector fields are important in differential equations:

3 A class of vector fields important in mechanics are **Hamiltonian fields**: If $H(x,y)$ is a function of two variables, then $[H_y(x,y), -H_x(x,y)]$ is called a **Hamiltonian vector field**. An example is the harmonic oscillator $H(x,y) = x^2 + y^2$. Its vector field $(H_y(x,y), -H_x(x,y)) = (y, -x)$. The flow lines of a Hamiltonian vector fields are located on the level curves of $H$.

4 Is the vector field $\vec{F}(x,y) = [P(x,y), Q(x,y)] = [xy, x^2]$ a gradient field?
**No.** Is the vector field $\vec{F}(x,y) = [P(x,y), Q(x,y)] = [\sin(x) + y, \cos(y) + x]$ a gradient field? Yes. the function is $f(x,y) = -\cos(x) + \sin(y) + xy$.

5 Can you spot the following vector fields in the pictures? $F(x,y) = [-y, 0]$, $F(x,y) = [-y - x, x + y]$, $F(x,y) = [y, -x]$, $F(x,y) = [y - x, x + y]$. Which ones are conservative?