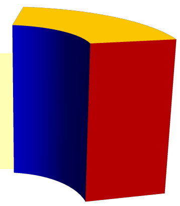
Lecture 24: Spherical integration

Cylindrical coordinates are coordinates in space in which polar coordinates are chosen in the xy-plane and where the z-coordinate is left untouched. A surface of revolution can be described in cylindrical coordinates as $r = g(z)$. The coordinate change transformation $T(r, \theta, z) = (r\cos(\theta), r\sin(\theta), z)$, produces the same integration factor $\boxed{r}$ as in polar coordinates.

$$\iint_{T(R)} f(x, y, z)\ dxdydz = \iint_R g(r, \theta, z)\ \boxed{\text{r}}\ drd\theta dz$$

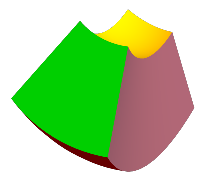


In spherical coordinates we use the distance ρ to the origin as well as the polar angle θ as well as ϕ, the angle between the vector and the z axis. The coordinate change is

$$T : (x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))\ .$$

It produces an integration factor is the volume of a **spherical wedge** which is $d\rho, \rho\sin(\phi)\ d\theta, \rho d\phi = \rho^2\sin(\phi)d\theta d\phi d\rho$.

$$\iint_{T(R)} f(x, y, z)\ dxdydz = \iint_R g(\rho, \theta, \phi)\ \boxed{\rho^2\sin(\phi)}\ d\rho d\theta d\phi$$



1 A sphere of radius R has the volume

$$\int_0^R \int_0^{2\pi} \int_0^\pi \rho^2\sin(\phi)\ d\phi d\theta d\rho\ .$$

The most inner integral $\int_0^\pi \rho^2\sin(\phi)d\phi = -\rho^2\cos(\phi)|_0^\pi = 2\rho^2$. The next layer is, because ϕ does not appear: $\int_0^{2\pi} 2\rho^2\ d\phi = 4\pi\rho^2$. The final integral is $\int_0^R 4\pi\rho^2\ d\rho = 4\pi R^3/3$.

The moment of inertia of a body G with respect to an z axes is defined as the triple integral $\int\int\int_G x^2 + y^2\ dzdydx$, where r is the distance from the axes.

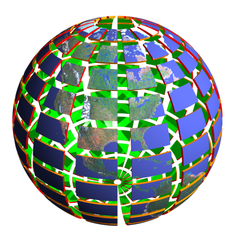
2 For a sphere of radius R we obtain with respect to the z-axis:

$$I = \int_0^R \int_0^{2\pi} \int_0^\pi \rho^2\sin^2(\phi)\rho^2\sin(\phi)\ d\phi d\theta d\rho$$

$$= (\int_0^\pi \sin^3(\phi)\ d\phi)(\int_0^R \rho^4\ dr)(\int_0^{2\pi}\ d\theta)$$

$$= (\int_0^\pi \sin(\phi)(1-\cos^2(\phi))\ d\phi)(\int_0^R \rho^4\ dr)(\int_0^{2pi}\ d\theta)$$

$$= (-\cos(\phi) + \cos(\phi)^3/3)|_0^\pi (L^5/5)(2\pi) = \frac{4}{3}\cdot\frac{R^5}{5}\cdot 2\pi = \frac{8\pi R^5}{15}\ .$$



If the sphere rotates with angular velocity ω, then $I\omega^2/2$ is the **kinetic energy** of that sphere. **Example:** the moment of inertia of the earth is $8 \cdot 10^{37} kgm^2$. The angular velocity is $\omega = 2\pi/day = 2\pi/(86400s)$. The rotational energy is $8 \cdot 10^{37} kgm^2/(7464960000s^2) \sim 10^{29} J \sim 2.510^{24} kcal$.

3 Find the volume and the center of mass of a diamond, the intersection of the unit sphere with the cone given in cylindrical coordinates as $z = \sqrt{3}r$.
Solution: we use spherical coordinates to find the center of mass

$$\begin{aligned}
\overline{x} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \sin^2(\phi) \cos(\theta)\ d\phi d\theta d\rho \frac{1}{V} = 0 \\
\overline{y} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \sin^2(\phi) \sin(\theta)\ d\phi d\theta d\rho \frac{1}{V} = 0 \\
\overline{z} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \cos(\phi) \sin(\phi)\ d\phi d\theta d\rho \frac{1}{V} = \frac{2\pi}{32V}
\end{aligned}$$

4 Find $\int\int\int_R z^2\ dV$ for the solid obtained by intersecting $\{1 \leq x^2 + y^2 + z^2 \leq 4\ \}$ with the double cone $\{z^2 \geq x^2 + y^2\}$.
Solution: since the result for the double cone is twice the result for the single cone, we work with the diamond shaped region R in $\{z > 0\}$ and multiply the result at the end with 2. In spherical coordinates, the solid R is given by $1 \leq \rho \leq 2$ and $0 \leq \phi \leq \pi/4$. With $z = \rho\cos(\phi)$, we have

$$\int_1^2 \int_0^{2\pi} \int_0^{\pi/4} \rho^4 \cos^2(\phi) \sin(\phi)\ d\phi d\theta d\rho$$

$$= (\frac{2^5}{5} - \frac{1^5}{5}) 2\pi (\frac{-\cos^3(\phi))}{3}|_0^{\pi/4} = 2\pi \frac{31}{5}(1 - 2^{-3/2})\ .$$

The result for the double cone is $\boxed{4\pi(31/5)(1 - 1/\sqrt{2}^3)}$.