Lecture 23: Triple integrals

If $f(x,y,z)$ is a function of three variables and E is a **solid region** in space, then $\int\int\int_E f(x,y,z)\; dxdydz$ is defined as the $n \to \infty$ limit of the Riemann sum

$$\frac{1}{n^3} \sum_{(i/n,j/n,k/n) \in E} f(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}) \; .$$

As in two dimensions, triple integrals can be evaluated by iterated 1D integral computations. Here is a simple example:

1 Assume E is the box $[0,1] \times [0,1] \times [0,1]$ and $f(x,y,z) = 24x^2y^3z$.

$$\int_0^1 \int_0^1 \int_0^1 24x^2y^3z \; dz \; dy \; dx \; .$$

To compute the integral we start from the core $\int_0^1 24x^2y^3z \; dz = 12x^3y^3$, then integrate the middle layer, $\int_0^1 12x^3y^3 \; dy = 3x^2$ and finally and finally handle the outer layer: $\int_0^1 3x^2 dx = 1$. When we calculate the most inner integral, we fix x and y. The integral is integrating up $f(x,y,z)$ along a line intersected with the body. After completing the middle integral, we have computed the integral on the plane $z = const$ intersected with R. The most outer integral sums up all these two dimensional sections.

The two important methods for triple integrals are the "washer method" and the "sandwich method". The washer method from single variable calculus reduces the problem directly to a one dimensional integral. The new sandwich method reduces the problem to a two dimensional integration problem.

The **washer method** slices the solid along the z-axes. If $g(z)$ is the double integral along the two dimensional slice, then $\int_a^b [\int\int_{R(z)} f(x,y,z) \; dxdy] \; dz$. The **sandwich method** sees the solid sandwiched between the graphs of two functions $g(x,y)$ and $h(x,y)$ over a common two dimensional region R. The integral becomes $\int\int_R [\int_{g(x,y)}^{h(x,y)} f(x,y,z) \; dz] \; dxdy$.

2 An important special case of the sandwich method is the volume

$$\int\int_R \int_0^{f(x,y)} 1 \; dzdxdy \; .$$

under the graph of a function $f(x,y)$ and above a region R. It is the integral $\int\int_R f(x,y) \; dA$. What we actually have computed is a triple integral

3 Find the volume of the unit sphere. **Solution:** The sphere is sandwiched between the graphs of two functions. Let R be the unit disc in the xy plane. If we use the **sandwich method**, we get

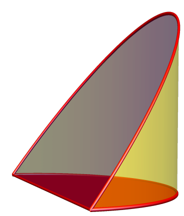
$$V = \int \int_R [\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1dz]dA \ .$$

which gives a double integral $\int \int_R 2\sqrt{1-x^2-y^2} \ dA$ which is of course best solved in polar coordinates. We have $\int_0^{2\pi} \int_0^1 \sqrt{1-r^2}r \ drd\theta = 4\pi/3$.
With the **washer method** which is in this case also called **disc method**, we slice along the z axes and get a disc of radius $\sqrt{1-z^2}$ with area $\pi(1-z^2)$. This is a method suitable for single variable calculus because we get directly $\int_{-1}^1 \pi(1-z^2) \ dz = 4\pi/3$.

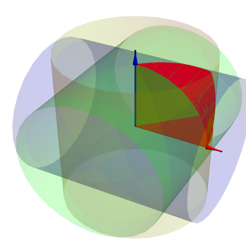
4 The mass of a body with density $\rho(x,y,z)$ is defined as $\int \int \int_R \rho(x,y,z) \ dV$. For bodies with constant density ρ the mass is ρV, where V is the volume. Compute the mass of a body which is bounded by the parabolic cylinder $z = 4 - x^2$, and the planes $x = 0, y = 0, y = 6, z = 0$ if the density of the body is 1. **Solution:**

$$\int_0^2 \int_0^6 \int_0^{4-x^2} \ dz \ dy \ dx = \int_0^2 \int_0^6 (4-x^2) \ dydx$$
$$= \ 6\int_0^2 (4-x^2) \ dx = 6(4x - x^3/3)|_0^2 = 32$$

5 The solid region bound by $x^2 + y^2 = 1$, $x = z$ and $z = 0$ is called the **hoof of Archimedes**. It is historically significant because it is one of the first examples, on which Archimedes probed his Riemann sum integration technique. It appears in every calculus text book. Find the volume. **Solution**. Look from the situation from above and picture it in the $x-y$ plane. You see a half disc R. It is the floor of the solid. The roof is the function $z = x$. We have to integrate $\int \int_R x \ dxdy$. We got a double integral problems which is best done in polar coordinates; $\int_{-\pi/2}^{\pi/2} \int_0^1 r^2 \cos(\theta) \ drd\theta = 2/3$.



6 Finding the volume of the solid region bound by the three cylinders $x^2+y^2 = 1$, $x^2+z^2 = 1$ and $y^2+z^2 = 1$ is one of the most famous volume integration problems. **Solution:** look at 1/16'th of the body given in cylindrical coordinates $0 \leq \theta \leq \pi/4, r \leq 1, z > 0$. The roof is $z = \sqrt{1-x^2}$ because above the "one eighth disc" R only the cylinder $x^2+z^2 = 1$ matters. The polar integration problem



$$16 \int_0^{\pi/4} \int_0^1 \sqrt{1-r^2\cos^2(\theta)}r \ drd\theta$$

has an inner r-integral of $(16/3)(1-\sin(\theta)^3)/\cos^2(\theta)$. Integrating this over θ can be done by integrating $(1 + \sin(x)^3)\sec^2(x)$ by parts using $\tan'(x) = \sec^2(x)$ leading to the anti derivative $\cos(x)+\sec(x)+\tan(x)$. The result is $16 - 8\sqrt{2}$.