# Lecture 22: Surface area

A surface $\vec{r}(u,v)$ parametrized on a parameter domain $R$ has the **surface area**

$$\int\int_R |\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \; dudv \; .$$

Proof. The vector $\vec{r}_u$ is tangent to the grid curve $u \mapsto \vec{r}(u,v)$ and $\vec{r}_v$ is tangent to $v \mapsto \vec{r}(u,v)$. The two vectors span a parallelogram with area $|\vec{r}_u \times \vec{r}_v|$. A small rectangle $[u, u+du] \times [v, v+dv]$ is mapped by $\vec{r}$ to a parallelogram spanned by $[\vec{r}, \vec{r} + \vec{r}_u]$ and $[\vec{r}, \vec{r} + \vec{r}_v]$ which has the area $|\vec{r}_u(u,v) \times \vec{r}_v(u,v)| \; dudv$.

1 The parametrized surface $\vec{r}(u,v) = [2u, 3v, 0]$ is part of the xy-plane. The parameter region $G$ just gets stretched by a factor 2 in the $x$ coordinate and by a factor 3 in the $y$ coordinate. $\vec{r}_u \times \vec{r}_v = [0,0,6]$ and we see for example that the area of $\vec{r}(G)$ is 6 times the area of $G$.

For a planar region $\vec{r}(s,t) = P + sv + tw$ where $(s,t) \in G$, the surface area is the area of $G$ times $|v \times w|$.

2 The map $\vec{r}(u,v) = [L\cos(u)\sin(v), L\sin(u)\sin(v), L\cos(v)]$ maps the rectangle $G = [0, 2\pi] \times [0, \pi]$ onto the sphere of radius $L$. We compute $\vec{r}_u \times \vec{r}_v = L\sin(v)\vec{r}(u,v)$. So, $|\vec{r}_u \times \vec{r}_v| = L^2|\sin(v)|$ and $\int\int_R 1 \; dS = \int_0^{2\pi} \int_0^{\pi} L^2 \sin(v) \; dvdu = 4\pi L^2$

For a sphere of radius $L$, we have $|\vec{r}_u \times \vec{r}_v| = L^2 \sin(v)$ The surface area is $4\pi L^2$.

3 For graphs $(u,v) \mapsto [u, v, f(u,v)]$, we have $\vec{r}_u = (1, 0, f_u(u,v))$ and $\vec{r}_v = (0, 1, f_v(u,v))$. The cross product $\vec{r}_u \times \vec{r}_v = (-f_u, -f_v, 1)$ has the length $\sqrt{1 + f_u^2 + f_v^2}$. The area of the surface above a region $G$ is $\int\int_G \sqrt{1 + f_u^2 + f_v^2} \; dudv$.

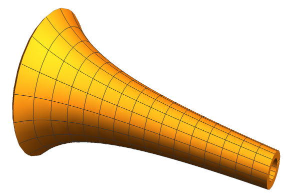
For a graph $z = f(x,y)$ parametrized over $G$, the surface area is

$$\int\int_G \sqrt{1 + f_x^2 + f_y^2} \; dxdy \; .$$

4 Lets take a surface of revolution $\vec{r}(u,v) = [v, f(v)\cos(u), f(v)\sin(u)]$ on $R = [0, 2\pi] \times [a,b]$. We have $\vec{r}_u = (0, -f(v)\sin(u), f(v)\cos(u)), \vec{r}_v = (1, f'(v)\cos(u), f'(v)\sin(u))$ and $\vec{r}_u \times \vec{r}_v = (-f(v)f'(v), f(v)\cos(u), f(v)\sin(u)) = f(v)(-f'(v), \cos(u), \sin(u))$. The surface area is $\int\int |\vec{r}_u \times \vec{r}_v| \; dudv = 2\pi \int_a^b |f(v)|\sqrt{1 + f'(v)^2} \; dv$.

For a surface of revolution $r = f(z)$ with $a \leq z \leq b$, the surface area is

$$2\pi \int_a^b |f(z)| \sqrt{1 + f'(z)^2} \, dz \; .$$



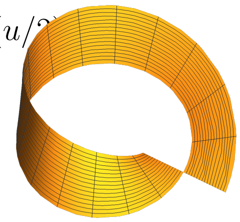
5 Gabriel's trumpet is the surface of revolution where $g(z) = 1/z$, where $1 \leq z < \infty$. Its volume is $\int_1^\infty \pi g(z)^2 \; dz = \pi$. We will compute in class the surface area.

6 Find the surface area of the part of the paraboloid $x = y^2 + z^2$ which is inside the cylinder $y^2 + z^2 \leq 9$. **Solution.** We use polar coordinates in the $yz$-plane. The paraboloid is parametrized by $(u, v) \mapsto (v^2, v\cos(u), v\sin(u))$ and the surface integral $\int_0^3 \int_0^{2\pi} |\vec{r}_u \times \vec{r}_v| \; dudv$ is equal to $\int_0^3 \int_0^{2\pi} v\sqrt{1+4v^2} \; dudv = 2\pi \int_0^3 v\sqrt{1+4v^2} \; dv = \pi(37^{3/2} - 1)/6$.

7 In this example we derive the distortion factor $r$ in polar coordinates. To do so, we parametrize a region in the $xy$ plane with $\vec{r}(u, v) = [u\cos(v), u\sin(v), 0]$. Given a region $G$ in the $uv$ plane like the rectangle $[0, \pi] \times [1, 2]$, we obtain a region $S$ in the $xy$ plane as the image. The factor $|\vec{r}_u \times \vec{r}_v|$ is equal to the radius $u$. In our example, the surface area is $\int_0^\pi \int_1^2 u \; dudv = \pi(4 - 1) = 3\pi$. This is the area of the half annulus $S$. We could have used polar coordinates directly in the $xy$ plane and compute $\int_0^\pi \int_1^2 r \; dr \; d\theta = 3\pi$. But the only thing which has changed are the names of the variables.

The surface parametrized by

$$\vec{r}(u, v) = [(2+v\cos(u/2))\cos(u), (2+v\cos(u/2))\sin(u), v\sin(u/2))]$$



8 on $G = [0, 2\pi] \times [-1, 1]$ is called a **Möbius strip**. What is its surface area? **Solution.** The calculation of $|\vec{r}_u \times \vec{r}_v|^2 = 4 + 3v^2/4 + 4v\cos(u/2) + v^2\cos(u)/2$ is straightforward but a bit tedious. The integral over $[0, 2\pi] \times [-1, 1]$ can only be evaluated numerically, the result is 25.413....