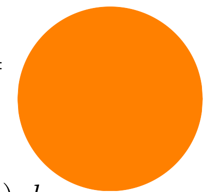
Lecture 21: Polar integration

1 The area of a disc of radius R is

$$\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 1\ dydx = \int_{-R}^{R} 2\sqrt{R^2-x^2}\ dx\ .$$



This integral can be solved with the substitution $x = R\sin(u), dx = R\cos(u)$

$$\int_{-\pi/2}^{\pi/2} 2\sqrt{R^2 - R^2\sin^2(u)} R\cos(u)\ du = \int_{-\pi/2}^{\pi/2} 2R^2\cos^2(u)\ du\ .$$

Using a double angle formula we get $R^2\int_{-\pi/2}^{\pi/2} 2\frac{(1+\cos(2u)}{2}\ du = R^2\pi$. We will now see how to do that better in polar coordinates.

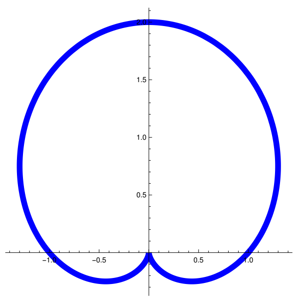
A **polar region** is a region bound by a simple closed curve given in polar coordinates as the curve $(r(t), \theta(t))$.

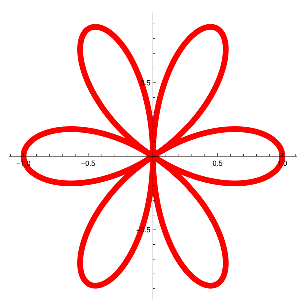
In Cartesian coordinates the parametrization of the boundary curve is $\vec{r}(t) = [r(t)\cos(\theta(t), r(t)\sin(\theta(t))]$. We are especially interested in regions which are bound by **polar graphs**, where $\theta(t) = t$.

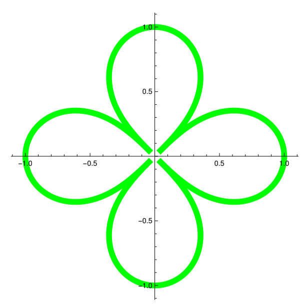
2 The **polar region** defined by $r \leq |\cos(3\theta)|$ belongs to the class of **roses** $r(t) = |\cos(nt)|$ they are also called **rhododenea**. These names reflect that polar regions model flowers well.

3 The polar curve $r(\theta) = 1 + \sin(\theta)$ is called a **cardioid**. It looks like a heart. It is a special case of a **limacon** a polar curve of the form $r(\theta) = 1 + b\sin(\theta)$.

4 The polar curve $r(\theta) = |\sqrt{\cos(2t)}|$ is called a **lemniscate**. It looks like an infinity sign. It encloses a flower with two petals.







To integrate in polar coordinates, we evaluate the integral

$$\int\int_R f(x,y)\ dxdy = \int\int_R f(r\cos(\theta), r\sin(\theta) r\ drd\theta$$

5 Integrate

$$f(x,y) = x^2 + x^2 + xy \ ,$$

over the unit disc. We have $f(x,y) = f(r\cos(\theta), r\sin(\theta)) = r^2 + r^2\cos(\theta)\sin(\theta)$ so that $\int\int_R f(x,y)\ dxdy = \int_0^1 \int_0^{2\pi} (r^2 + r^2\cos(\theta)\sin(\theta))r\ d\theta dr = 2\pi/4$.

6 We have earlier computed area of the disc $\{x^2 + y^2 \leq R^2\ \}$ using substitution. It is more elegant to do this integral in polar coordinates: $\int_0^{2\pi} \int_0^R r\ drd\theta = 2\pi r^2/2|_0^R = \pi R^2$.

Why do we have to include the factor r, when we move to polar coordinates? The reason is that a small rectangle R with dimensions $d\theta dr$ in the (r,θ) plane is mapped by $T : (r,\theta) \mapsto (r\cos(\theta), r\sin(\theta))$ to a sector segment S in the (x,y) plane. It has the area $r\ d\theta dr$.

7 Integrate the function $f(x,y) = 1\ \{(\theta, r(\theta))\ |\ r(\theta) \leq\ |\ \cos(3\theta)|\ \}$.

$$\int\int_R 1\ dxdy = \int_0^{2\pi} \int_0^{\cos(3\theta)} r\ dr\ d\theta = \int_0^{2\pi} \frac{\cos(3\theta)^2}{2}\ d\theta = \pi/2\ .$$

8 Integrate $f(x,y) = y\sqrt{x^2+y^2}$ over the region $R = \{(x,y)\ |\ 1 < x^2 + y^2 < 4, y > 0\ \}$.

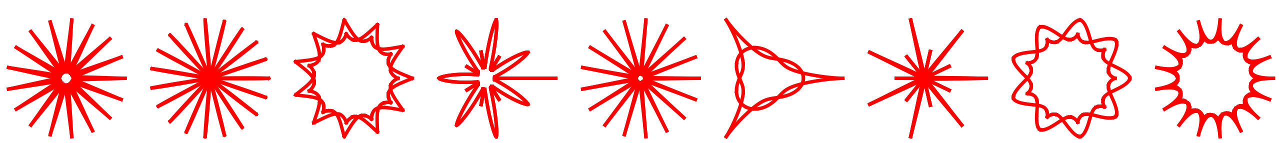
$$\int_1^2 \int_0^{\pi} r\sin(\theta) r\ r\ d\theta dr = \int_1^2 r^3 \int_0^{\pi} \sin(\theta)\ d\theta dr = \frac{(2^4 - 1^4)}{4} \int_0^{\pi} \sin(\theta)\ d\theta = 15/2$$

For integration problems, where the region is part of an annular region, or if you see function with terms $x^2 + y^2$ try to use polar coordinates $x = r\cos(\theta), y = r\sin(\theta)$.

9 The Belgian Biologist **Johan Gielis** defined in 1997 with the family of curves given in polar coordinates as

$$r(\phi) = \left(\frac{|\cos(\frac{m\phi}{4})|^{n_1}}{a} + \frac{|\sin(\frac{m\phi}{4})|^{n_2}}{b}\right)^{-1/n_3}$$

This **super-curve** can produce a variety of shapes like circles, square, triangle, stars. It can also be used to produce "super-shapes". The super-curve generalizes the **super-ellipse** which had been discussed in 1818 by Lamé and helps to **describe forms** in biology. [1]



[1] "Gielis, J. A 'generic geometric transformation that unifies a wide range of natural and abstract shapes'. American Journal of Botany, 90, 333 - 338, (2003).