Lecture 20: Double integrals

The integral $\int\int_R f(x,y)\ dxdy$ is defined as the limit of Riemann sums

$$\frac{1}{n^2} \sum_{(\frac{i}{n},\frac{j}{n})\in R} f(\frac{i}{n},\frac{j}{n})$$

when $n \to \infty$.

1 In order to integrate $f(x,y) = xy$ over the unit square, we can sum up the Riemann sum for fixed $y = j/n$ and get $y/2$. Integrating with respect to y from 0 to 1 gives $1/4$. This example shows how we can reduce double integrals to single variable integrals.

2 If $f(x,y) = 1$, then the integral is the **area** of the region R. The integral is the limit $L(n)/n^2$, where $L(n)$ is the number of lattice points $(i/n, j/n)$ inside R.

3 The integral $\iint_R f(x,y)\ dxdy$ as the **signed volume** of the solid below the graph of f and above the region R in the $x-y$ plane. The volume below the xy-plane is counted negatively.

Fubini's theorem allows to switch the order of integration over a rectangle, if the function f is continuous: $\int_a^b \int_c^d f(x,y)\ dxdy = \int_c^d \int_a^b f(x,y)\ dydx$.

Proof. For every n, check the "quantum Fubini identity"

$$\sum_{\frac{i}{n}\in[a,b]} \sum_{\frac{j}{n}\in[c,d]} f(\frac{i}{n},\frac{j}{n}) = \sum_{\frac{j}{n}\in[c,d]} \sum_{\frac{i}{n}\in[a,b]} f(\frac{i}{n},\frac{j}{n})$$

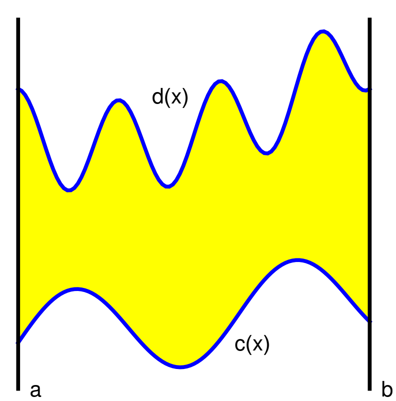
holds for all functions. Now divide both sides by n^2 and take the limit $n \to \infty$.

A **dy dx region** is of the form

$$R = \{(x,y) \mid a \le x \le b,\ c(x) \le y \le d(x)\ \}\ .$$

An integral over such a region is called a **dy dx integral**

$$\iint_R f\ dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)\ dydx\ .$$

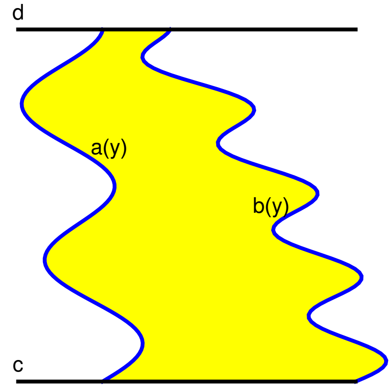


A **dx dy region** is of the form

$$R = \{(x,y) \mid c \leq y \leq d, a(y) \leq x \leq b(y)\ \} \ .$$

An integral over such a region is called a **dx dy integral**

$$\int\int_R f\ dA = \int_c^d \int_{a(y)}^{b(y)} f(x,y)\ dxdy \ .$$



4 Integrate $f(x,y) = x^2$ over the region bounded above by $\sin(x^3)$ and bounded below by the graph of $-\sin(x^3)$ for $0 \leq x \leq \pi$. The value of this integral has a physical meaning. It is called **moment of inertia**.

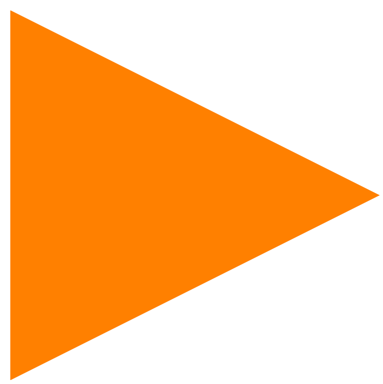
$$\int_0^{\pi^{1/3}} \int_{-\sin(x^3)}^{\sin(x^3)} x^2\ dydx = 2\int_0^{\pi^{1/3}} \sin(x^3)x^2\ dx$$

This can be solved by substitution

$$= -\frac{2}{3}\cos(x^3)|_0^{\pi^{1/3}} = \frac{4}{3} \ .$$

5 Integrate $f(x,y) = y^2$ over the region bound by the x-axes, the lines $y = x+1$ and $y = 1-x$. The problem is best solved as a dy dx integral. Because we would have to compute 2 different integrals as a dy dx integral. The y bounds are $x = y-1$ and $x = 1-y$

$$\int_0^1 \int_{y-1}^{1-y} y^3\ dx\ dy = 2\int_0^1 y^3(1-y)\ dy = 2(\frac{1}{4}-\frac{1}{3}) = \frac{1}{10} \ .$$



6 Let R be the triangle $1 \geq x \geq 0, 0 \leq y \leq x$. What is

$$\int\int_R e^{-x^2}\ dxdy \ ?$$

The dx dy integral $\int_0^1[\int_y^1 e^{-x^2}\ dx]dy$ can not be solved because e^{-x^2} has no anti-derivative in terms of elementary functions. The dy dx integral $\int_0^1[\int_0^x e^{-x^2}\ dy]\ dx$ however can be solved:

$$= \int_0^1 xe^{-x^2}\ dx = -\frac{e^{-x^2}}{2}|_0^1 = \frac{(1-e^{-1})}{2} = 0.316... \ .$$