19: Global Extrema

To determine the maximum or minimum of $f(x,y)$ on a region, we find first all critical points **in the interior the domain**, then compute all critical points **at the boundary**. This involves to solve an extremal problems with constraints and one without constraints. The largest value among all critical values leads to the maximum.

A point (a,b) is called a **global maximum** of $f(x,y)$ on a region G $f(x,y) \leq f(a,b)$ for all (x,y) in G. If G is not specified, we assume $G = \mathbb{R}^2$. For example, the point $(0,0)$ is a global maximum of the function $f(x,y) = 1 - x^2 - y^2$. Similarly, we call (a,b) a **global minimum**, if $f(x,y) \geq f(a,b)$ for all (x,y).

1 **Question:** Does the function $f(x,y) = x^4 + y^4 - 2x^2 - 2y^2$ have a global maximum or a global minimum on $\mathbb{R}^2$? If yes, find them. **Solution:** the function has no global maximum on $\mathbb{R}^2$. This can be seen by restricting the function to the x-axis, where $f(x,0) = x^4 - 2x^2$ is a function without maximum. The function has four global minima however. They are located on the 4 points $(\pm 1, \pm 1)$. The best way to see this is to note that $f(x,y) = (x^2-1)^2 + (y-1)^2 - 2$ which is minimal when $x^2 = 1, y^2 = 1$.

2 Find the maximum of $f(x,y) = 2x^2 - x^3 - y^2$ on the region $y \geq -1$. **Solution.** With $\nabla f(x,y) = 4x - 3x^2, -2y)$, the critical points are $(4/3, 0)$ and $(0,0)$. The Hessian is $H(x,y) = \begin{bmatrix} 4-6x & 0 \\ 0 & -2 \end{bmatrix}$. At $(0,0)$, the discriminant is -8 so that this is a saddle point. At $(4/3,0)$, the discriminant is 8 and $H_{11} = 4/3$, so that $(4/3,0)$ is a local maximum. We have now also to look at the boundary $y = -1$ where the function is $g(x) = f(x,-1) = 2x^2 - x^3 - 1$. Since $g'(x) = 0$ at $x = 0, 4/3$, where 0 is a local minimum, and $4/3$ is a local maximum on the line $y = -1$. Comparing $f(4/3,0), f(4/3,-1)$ shows that $(4/3,0)$ is the global maximum.

3 Find all extrema of the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$ on the plane and characterize them. Do you find a global maximum or global minimum among them? **Solution.** The critical points satisfy $\nabla f(x,y) = [0,0]$ or $[3x^2 - 3, 3y^2 - 12] = [0,0]$. There are 4 critical points $(x,y) = (\pm 1, \pm 2)$. The discriminant is $D = f_{xx} f_{yy} - f_{xy}^2 = 36xy$ and $f_{xx} = 6x$.

point	D	f_{xx}	classification	value
(-1,-2)	72	-6	maximum	38
(-1, 2)	-72	-6	saddle	6
(1, -2)	-72	6	saddle	34
(1, 2)	72	6	minimum	2

There are no global maxima nor global minima because the function takes arbitrarily large and small values. For $y = 0$ the function is $g(x) = f(x,0) = x^3 - 3x + 20$ which satisfies $\lim_{x \to \pm\infty} g(x) = \pm\infty$.

You can ignore the following questions and answers if you like.

1. **Do global extrema always exist?** Yes, if the region Y is **compact** meaning that for every sequence x_n, y_n we can pick a subsequence which converges in Y. This is equivalent that the domain is **closed and bounded**.

Bolzano's extremal value theorem. Every continuous function on a compact domain has a global maximum and a global minimum.

2. **Why are critical points important?** Critical points are relevant in physics because they represent configurations with lowest energy. Many physical laws describe extrema. The Newton equations $mr''(t)/2 - \nabla V(r(t)) = 0$ describing a particle of mass m moving in a field V along a path $\gamma : t \mapsto \vec{r}(t)$ are equivalent to the property that the path extremizes the are length $S(\gamma) = \int_a^b mr'(t)^2/2 - V(r(t))\, dt$ among all paths.

3. **Why is the second derivative test true?** Assume $f(x,y)$ has the critical point $(0,0)$ and is a quadratic function satisfying $f(0,0) = 0$. Then $ax^2 + 2bxy + cy^2 = a(x + \frac{b}{a}y)^2 + (c - \frac{b^2}{a})y^2 = a(A^2 + DB^2)$ with $A = (x + \frac{b}{a}y), B = b^2/a^2$ and discriminant D. You see that if $a = f_{xx} > 0$ and $D > 0$ then $c - b^2/a > 0$ and the function has positive values for all $(x,y) \neq (0,0)$. The point $(0,0)$ is a minimum. If $a = f_{xx} < 0$ and $D > 0$, then $c - b^2/a < 0$ and the function has negative values for all $(x,y) \neq (0,0)$ and the point (x,y) is a local maximum. If $D < 0$, then f takes both negative and positive values near $(0,0)$. For a general function approximate by a quadratic one.

4. **Is there something cool about critical points?** Yes, assume $f(x,y)$ be the height of an island. Assume there are only finitely many critical points and all of them have nonzero determinant. Label each critical point with a $+1$ if it is a maximum or minimum, and with -1 if it is a saddle point. The sum of all these numbers is 1, independent of the island. [1]

5) **Can we avoid Lagrange by substitution?** To extremize $f(x,y)$ under the constraint $g(x,y) = 0$ we find $y = y(x)$ from the second equation and extremize the single variable problem $f(x, y(x))$. To extremize $f(x,y) = y$ on $x^2 + y^2 = 1$ for example we need to extremize $\sqrt{1 - x^2}$. We can differentiate to get the critical points but also have to look at the cases $x = 1$ and $x = -1$, where the actual minima and maxima occur. In general also, we can not do the substitution.

6) **Is there a second derivative test for Lagrange?** A second derivative test can be designed using second directional derivative in the direction of the tangent. Instead, we just make a list of critical points and pick the maximum and minimum.

7) **Does Lagrange also work with more constraints?** With two constraints the constraint $g = c, h = d$ defines a curve. The gradient of f must now be in the plane spanned by the gradients of g and h because otherwise, we could move along the curve and increase f. Here is a formulation in three dimensions. Extrema of $f(x,y,z)$ under the constraint $g(x,y,z) = c, h(x,y,z) = d$ are either solutions of the Lagrange equations $\nabla f = \lambda \nabla g + \mu \nabla h, g = c, h = d$ or solutions to $\nabla g = 0, \nabla f(x,y,z) = \mu \nabla h, h = d$ or solutions to $\nabla h = 0, \nabla f = \lambda \nabla g, g = c$ or solutions to $\nabla g = \nabla h = 0$.

8) **Why do D and f_{xx} appear in the second derivative test**. They are natural. The discriminant D is a determinant $\det(H)$ of the matrix $H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$. If $D > 0$ then the sign of f_{xx} is the same as the sign of the trace $f_{xx} + f_{yy}$ which is coordinate independent too. The determinant is the product $\lambda_1 \lambda_2$ of the eigenvalues of H and the trace is the sum of the eigenvalues.

9) **What does D mean?** The discriminant D is defined also at points where we have no critical point. The number $K = D/(1 + |\nabla f|^2)^2$ is called the **Gaussian curvature** of the surface. At critical points $K = D$. Curvature is remarkable quantity since it only depends on the intrinsic geometry of the surface and not on the way how the surface is embedded in space. [2]

10) **Is there a 2. derivative test in higher dimensions?** Yes. one can form the second derivative matrix H and look at all the eigenvalues of H. If all eigenvalues are negative, we have a local maximum, if all eigenvalues are positive, we have a local minimum. In general eigenvalues have different signs and we have a saddle point type.

[1] This follows from the **Poincare-Hopf** theorem.

[2] This is the **Theorema Egregia of Gauss**.