16: Directional Derivative

If f is a function of several variables and $\vec{v}$ is a unit vector, then

$$D_{\vec{v}} f = \nabla f \cdot \vec{v}$$

is called the **directional derivative** of f in the direction $\vec{v}$.

The name directional derivative is related to the fact that unit vectors are directions. Because of the chain rule $\frac{d}{dt} D_{\vec{v}} f = \frac{d}{dt} f(x + t\vec{v})$, the directional derivative tells us how the function changes when we move in a given direction. Assume for example that $f(x, y, z)$ is the temperature at position (x, y, z). If we move with velocity $\vec{v}$ through space, then $D_{\vec{v}} f$ tells us at which rate the temperature changes for us. If we move with velocity $\vec{v}$ on a hilly surface of height $f(x, y)$, then $D_{\vec{v}} f(x, y)$ gives us the slope in the direction $\vec{v}$.

1 If $\vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$ and the speed is 1, then $D_{\vec{r}'(t)} f = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$ is the temperature change, one measures at $\vec{r}(t)$. The chain rule told us that this is $\frac{d}{dt} f(\vec{r}(t))$.

2 For $\vec{v} = [1, 0, 0]$, then $D_{\vec{v}} f = \nabla f \cdot v = f_x$. The directional derivative generalizes the partial derivatives. It measures the rate of change of f, if we walk with unit speed into that direction. But as with partial derivatives, it is a **scalar**.

The directional derivative satisfies $\boxed{|D_{\vec{v}} f| \leq |\nabla f|}$.

Proof. $\nabla f \cdot \vec{v} = |\nabla f||\vec{v}|| \cos(\phi)| \leq |\nabla f||\vec{v}|$.
This implies

The gradient points in the direction where f increases most.

At a point where the gradient ∇f is not the zero vector, the direction $\vec{v} = \nabla f / |\nabla f|$ is the direction, where f **increases** most. It is the direction of **steepest ascent**.

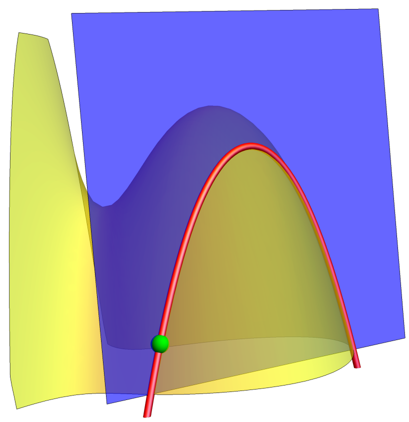
If $\vec{v} = \nabla f / |\nabla f|$, then the directional derivative is $\nabla f \cdot \nabla f / |\nabla f| = |\nabla f|$. This means f **increases**, if we move into the direction of the gradient. The slope in that direction is $|\nabla f|$.

3 You are in an airship at $(1, 2)$ and want to avoid a thunderstorm, a region of low pressure, where pressure is $p(x, y) = x^2 + 2y^2$. In which direction do you have to fly so that the pressure decreases fastest? **Solution:** the pressure gradient is $\nabla p(x, y) = [2x, 4y]$. At the point $(1, 2)$ this is $[2, 8]$. Normalize to get the direction $\vec{v} = [1, 4]/\sqrt{17}$. If you want to head into the direction where pressure is lower, go towards $-\vec{v}$.

Directional derivatives satisfy the same properties then any derivative: $D_v(\lambda f) = \lambda D_v(f)$, $D_v(f + g) = D_v(f) + D_v(g)$ and $D_v(fg) = D_v(f)g + f D_v(g)$.

We will see later that points with $\nabla f = \vec{0}$ are candidates for **local maxima** or **minima** of f. Points (x, y), where $\nabla f(x, y) = [0, 0]$ are called **critical points** and help to understand the function f.

4 Problem. Assume we know $D_v f(1,1) = 3/\sqrt{5}$ and $D_w f(1,1) = 5/\sqrt{5}$, where $v = [1,2]/\sqrt{5}$ and $w = [2,1]/\sqrt{5}$. Find the gradient of f. Note that we do not know anything else about the function f.
Solution: Let $\nabla f(1,1) = [a, b]$. We know $a + 2b = 3$ and $2a + b = 5$. This allows us to get $a = 7/3, b = 1/3$.



If you should be interested in higher derivatives. We have seen that we can compute f_{xx}. This can be seen as the second directional derivative in the direction $(1, 0)$.

5 The Matterhorn is a famous mountain in the Swiss alps. Its height is 4'478 meters (14'869 feet). Assume in suitable units on the ground, the height $f(x, y)$ of the Matterhorn is approximated by $f(x, y) = 4000 - x^2 - y^2$. At height $f(-10, 10) = 3800$, at the point $(-10, 10, 3800)$, you rest. The climbing route continues into the south-east direction $\vec{v} = (1, -1)/\sqrt{2}$. Calculate the rate of change in that direction.
We have $\nabla f(x, y) = [-2x, -2y]$, so that $(20, -20) \cdot (1, -1)/\sqrt{2} = 40/\sqrt{2}$. This is a place, where you climb $40/\sqrt{2}$ meters up when advancing 1 meter forward.

We can also look at higher derivatives in a direction. It can be used to measure the concavity of the function in the $\vec{v}$ direction.

The second directional derivative in the direction $\vec{v}$ is $D_{\vec{v}} D_{\vec{v}} f(x, y)$.

6 For the function $f(x, y) = x^2 + y^2$ the first directional derivative at a point in the direction $[1, 2]/\sqrt{5}$ is $[2x, 2y] \cdot [1, 2] = (2x + 4y)/\sqrt{5}$. The second directional derivative in the same direction is $[2, 4] \cdot [1, 2]/5 = 6/5$. This reflects the fact that the graph of f is concave up in the direction $[1, 2]/5$.