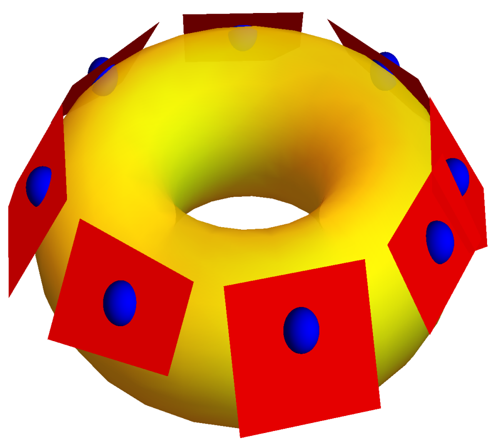
15: Gradient and Tangent

The **gradient** $\nabla f(x,y) = [f_x(x,y), f_y(x,y)]$ or $\nabla f(x,y,z) = [f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)]$ in three dimensions is an important object in multi variable calculus. It is the analog of the derivative $f'(x)$ in one dimensions. The symbol ∇ is spelled "Nabla" and named after an Egyptian harp. The following theorem is important because it provides a crucial link between calculus and geometry. It holds both in two and three dimensions:

Gradient theorem: Gradients are orthogonal to level curves or level surfaces respectively.

Proof: Every curve $\vec{r}(t)$ on the level curve or level surface satisfies $\frac{d}{dt} f(\vec{r}(t)) = 0$. By the chain rule, $\nabla f(\vec{r}(t))$ is perpendicular to the tangent vector $\vec{r}'(t)$. Because this is true for every curve, the gradient is perpendicular to the surface.

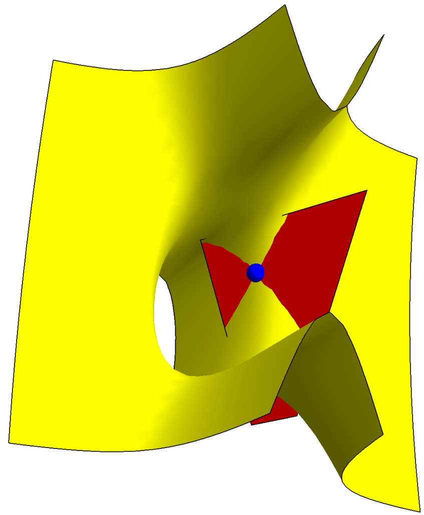


The gradient theorem is useful for example because it allows to get tangent planes and tangent lines very fast, faster than by making a linear approximation:

The tangent plane through $P = (x_0, y_0, z_0)$ to a level surface of $f(x,y,z)$ is $ax + by + cz = d$, where $\nabla f(x_0, y_0, z_0) = [a, b, c]$ and d is obtained by plugging in the point P.

The statement in two dimensions is analog.

1 Find the tangent plane to the surface $3x^2y + z^2 - 4 = 0$ at the point $(1,1,1)$. **Solution:** $\nabla f(x,y,z) = [6xy, 3x^2, 2z]$. And $\nabla f(1,1,1) = [6,3,2]$. The plane is $6x+3y+2z = d$ where d is a constant. We can find the constant d by plugging in a point and get $6x+3y+2z = 11$.



2 **Problem:** Find the tangent line to the curve $f(x,y) = x^3 - y^2x = 3$ at the point $(-1,2)$. **Solution:** The gradient is $\nabla f(x,y) = [3x^2 - y^2, -2yx]$ and $\nabla f(-1,2) = [-1,4]$. The tangent line is $-x + 4y = d$. We get the constant by plugging in the point $(-1,2)$. It is 9. The line is $-x + 4y = 9$.