11: Partial derivatives

If $f(x,y)$ is a function of two variables, then $\frac{\partial}{\partial x} f(x,y)$ is defined as the derivative of the function $g(x) = f(x,y)$, where y is considered a constant. It is called **partial derivative** of f with respect to x. The partial derivative with respect to y is defined similarly.

We also write $f_x(x,y) = \frac{\partial}{\partial x} f(x,y)$. and $f_{yx} = \frac{\partial}{\partial x}\frac{\partial}{\partial y} f$. [1]

1 For $f(x,y) = x^4 - 6x^2y^2 + y^4$, we have $f_x(x,y) = 4x^3 - 12xy^2$, $f_{xx} = 12x^2 - 12y^2$, $f_y(x,y) = -12x^2y + 4y^3$, $f_{yy} = -12x^2 + 12y^2$ and see that $f_{xx} + f_{yy} = 0$. A function which satisfies this equation is also called **harmonic**. The equation $f_{xx} + f_{yy} = 0$ is an example of a **partial differential equation** for the unknown function $f(x,y)$ involving partial derivatives. The vector $[f_x, f_y]$ is called the gradient.

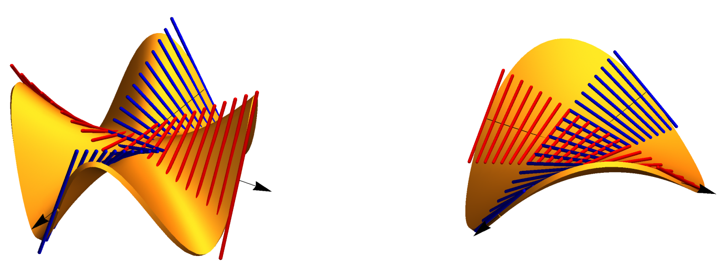
Clairaut's theorem If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$.

Proof: we look at the equations without taking limits first. We extend the definition and say that a background Planck constant h is positive, then $f_x(x,y) = [f(x+h,y) - f(x,y)]/h$. For $h = 0$ we define f_x as before. Compare the two sides for fixed $h > 0$:

$hf_x(x,y) = f(x+h,y) - f(x,y)$
$h^2 f_{xy}(x,y) = f(x+h,y+h) - f(x+h,y+h) - (f(x+h,y) - f(x,y))$

$hf_y(x,y) = f(x,y+h) - f(x,y)$.
$h^2 f_{yx}(x,y) = f(x+h,y+h) - f(x+h,y) - (f(x,y+h) - f(x,y))$

No limits were taken. We established an identity which holds for all $h > 0$, the discrete derivatives f_x, f_y satisfy $f_{xy} = f_{yx}$. It is a "quantum Clairaut" theorem. If the classical derivatives f_{xy}, f_{yx} are both continuous, the limit $h \to 0$ leads to the classical Clairaut's theorem. The quantum Clairaut theorem holds for **any** functions $f(x,y)$ of two variables. Not even continuity is needed.



2 Find $f_{xxxxxyxxxxx}$ for $f(x) = \sin(x) + x^6 y^{10} \cos(y)$. Hint: No need not compute, just think.

3 The continuity assumption for f_{xy} is necessary. The example $f(x,y) = \frac{x^3y - xy^3}{x^2+y^2}$ contradicts Clairaut's theorem:

[1] $\partial_x f, \partial_y f$ were introduced by Carl Gustav Jacobi. Josef Lagrange had used the term "partial differences".

$f_x(x,y) = (3x^2y - y^3)/(x^2+y^2) - 2x(x^3y - xy^3)/(x^2+y^2)^2$, $f_x(0,y) = -y$, $f_{xy}(0,0) = -1$, $f_y(x,y) = (x^3 - 3xy^2)/(x^2+y^2) - 2y(x^3y - xy^3)/(x^2+y^2)^2$, $f_y(x,0) = x$, $f_{y,x}(0,0) = 1$.

$f_x(x_0, y_0)$ measures the slope when slicing the graph $z = f(x,y)$ in the x-direction.
f_{xx} measures the concavity when slicing the graph in the x-direction.
f_{xy} measures how the x slope changes when you move in the y direction.

An equation for an unknown function $f(x,y)$ which involves partial derivatives with respect to at least two different variables is called a **partial differential equation**. If only the derivative with respect to one variable appears, it is called an **ordinary differential equation**.

Here are two examples of partial differential equations. We will look at them in more detail next time and try to make sense what they mean.

4 The **wave equation** $f_{tt}(t,x) = f_{xx}(t,x)$ governs the motion of light or sound. The function $f(t,x) = \sin(x-t) + \sin(x+t)$ satisfies the wave equation.

5 The **heat equation** $f_t(t,x) = f_{xx}(t,x)$ describes diffusion of heat or spread of an epidemic. The function $f(t,x) = \frac{1}{\sqrt{t}} e^{-x^2/(4t)}$ satisfies the heat equation.

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8 The **Laplace equation** $f_{xx} + f_{yy} = 0$ determines the shape of a membrane. The function $f(x,y) = x^3 - 3xy^2$ is an example satisfying the Laplace equation. Such functions are called **harmonic**.

9 The **advection equation** $f_t = f_x$ is used to model transport in a wire. The function $f(t,x) = e^{-(x+t)^2}$ satisfy the advection equation.

10 The **Burgers equation** $f_t + f f_x = f_{xx}$ describes waves at the beach which break. The function $f(t,x) = \frac{x}{t} \frac{\sqrt{\frac{1}{t}} e^{-x^2/(4t)}}{1 + \sqrt{\frac{1}{t}} e^{-x^2/(4t)}}$ satisfies the Burgers equation.