10: Functions

A function $f(x,y)$ with domain R is called **continuous at a point** $(a,b) \in R$ if $f(x,y) \to f(a,b)$ whenever $(x,y) \to (a,b)$. The function f is **continuous on** R, if f is continuous for every point (a,b) on R. Sometimes, we can extend the domain R to make it continuous there too. The function $f(x,y) = \sin(x)y/x$ for example can be assigned the value y on the axes $x = 0$ as l'Hopital shows.

1 The function $f(x,y) = x^2 + y^4 + xy + \sin(y + \sin\sin\sin\sin(x)^2)$ is continuous on the entire plane. It is built up from functions which are continuous using addition, multiplication or composition of functions which are all continuous.

2 $f(x,y) = 1/(x^2 + y^2)$ is continuous everywhere except at the origin, where it is not defined. We can not extend the value as the value at $(0,0)$ would have to be arbitrarily large.

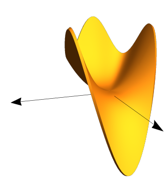
3 $f(x,y) = y + \sin(x)/|x|$ is continuous except at $x = 0$. At every point $(0,y)$ it is discontinuous. $f(1/n,y) \to y+1$ and $f(-1/n,y) \to y-1$ for $n \to \infty$.

4 $f(x,y) = \sin(1/(x+y))$ is continuous except on the line $x + y = 0$.

5

$$f(x,y) = (x^4 - y^5)/(x^2 + y^2)$$

is continuous at $(0,0)$. Use polar coordinates to see that it is $r^2\cos^2(t)$.



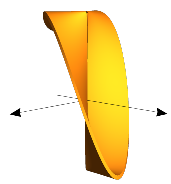
6 The function $f(x) = e^{-1/x^2}$ is continuous everywhere. Actually, even all derivatives are continuous and are zero at 0. Still, the function is not constant zero.

7 There are three sources for discontinuous behavior: there can be **jumps**, there can be **poles**, or the function can **oscillate**. An example of a jump appears with $f(x) = \sin(x)/|x|$, a pole example is $g(x) = 1/x$ leads to a vertical asymptote and the function going to infinity. An example of a function discontinuous due to oscillations is $h(x) = \sin(1/x)$. Its graph is the **devil's comb**.

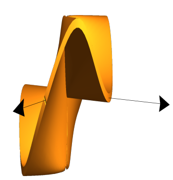
There are two handy tools to discover a discontinuities:
1) Use polar coordinates with coordinate center at the point to analyze the function.
2) Restrict the function to one dimensional curves and check continuity on that curve, where one has a function of one variables.

8 Determine whether the function $f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$ is continuous at $(0,0)$. **Solution** Use polar coordinates to write this as $\sin(r^2)/r^2$ which is continuous at 0 (apply l'Hopital twice if you want to verify this).

9 Is the function $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ continuous at $(0,0)$? **Solution** Use polar coordinates to see that this $\cos(2\theta)$. We see that the value depends on the angle only. Arbitrarily close to $(0,0)$, the function takes any value from -1 to 1.



10 Is the function $f(x,y) = \frac{x^2y}{x^4+y^2}$ continuous?
Solution. Look on the parabola $x^2 = y$ to get the function $x^4/(2x^4) = 1/(2x^2)$ which is not continuous at 0. This example is **shocking** because it is continuous through each line through the origin: if $y = ax$, then $f(x,ax) = ax^3/(x^4 + a^2x^2) = ax/(x^2 + a^2)$. This converges to 0 for $x \to 0$ as long as $a \neq 0$. If $a = 0$ however, we have $y = 0$ and $f = 0/x^4$ which can be continuously extended to $x = 0$ too.



11 What about the function

$$f(x,y) = \frac{xy^2 + y^3}{x^2 + y^2}$$

Solution. Use polar coordinates and write $r^3 \sin^2(\theta)(\cos(\theta) + \sin(\theta)/r^2 = r \sin^2(\theta)(\cos(\theta) + \sin(\theta)$ which shows that the function converges to 0 as $r \to 0$.

12 Is the function $f(x,y) = \frac{\sin(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}$ continuous everywhere? **Solution.** Use polar coordinates to see that this is $\sin(r)/r$. This function is continuous at 0 by Hôpital's theorem.