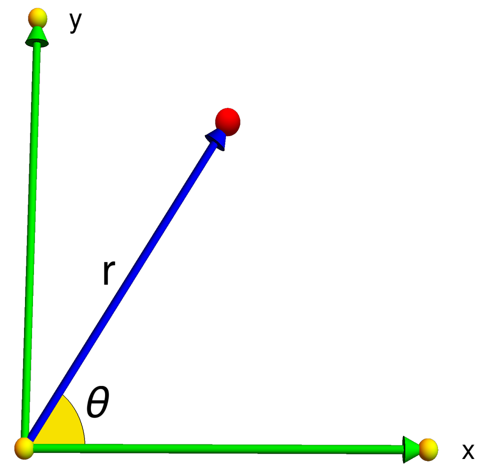
7: Polar and cylindrical coordinates

We first look at polar coordinates in two dimensions:

A point (x, y) in the plane has the **polar coordinates** $r = \sqrt{x^2 + y^2}, \theta = \arctan(y/x)$ leading to the relation $(x, y) = (r\cos(\theta), r\sin(\theta))$.

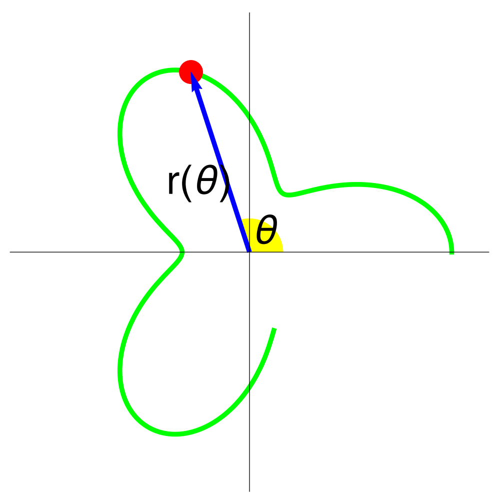


The formula $\theta = \arctan(y/x)$ defines the angle θ only up to an addition of π as (x, y) and $(-x, -y)$ have the same θ value. It is custom to let $\arctan(y/x)$ be in $(-\pi/2, \pi/2]$ for $x > 0$ and define it to be $\pi/2$ for the positive y axes and $\arctan(y/x) + \pi$ for $x < 0$ and equal to $-\pi/2$ on the negative y-axes. For $(x, y) = (0, 0)$, the polar angle θ is not defined.

Some curves can be described nicely in polar coordinates. For example, the unit circle is

$$r = 1 \ .$$

A curve given in polar coordinates as $r(\theta) = f(\theta)$ is called a **polar curve**. It can in Cartesian coordinates be described as $\vec{r}(t) = [f(t)\cos(t), f(t)\sin(t)]$.



1 The curve

$$\vec{r}(t) = [t\cos(t), t\sin(t)] = [x(t), y(t)]$$

describes a spiral. How can one describe this curve in polar coordinates? We see that $t = \theta$ is the angle and that the distance to the origin is $t = \theta$. Therefore, the curve is

$$r = \theta$$

2 What is the curve given in polar coordinates as

$$r = |2\sin(\theta)| .$$

Solution: Let us ignore the absolute value for a moment and multiply both sides with r. This gives

$$r^2 = 2r\sin(\theta)$$

and can be written as $x^2+y^2 = 2y$ which is $x^2+y^2-2y+1 = 1$. A completion of the square shows that this curve is a circle of radius 1 centered at $(0,1)$. Since we have the absolute value, we get an other circle of radius 1 centered at $(0,-1)$. This is when θ is between π and 2π.

Writing a point

$$(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$$

in the form

$$(r, \theta, z)$$

means using **cylindrical coordinates**. It is just using polar coordinates in the xy-plane and keeping the variable z.

Here are some surfaces described in cylindrical coordinates:

3 $r = 1$ is a **cylinder**,

4 $r = |z|$ is a **double cone**

5 $\theta = 0$ is a **half plane**

6 $r = \theta$ is a **rolled sheet of paper**

7 $r = 2 + \sin(z)$ is an example of a **surface of revolution**.

Remark which is not relevant for this course but which will come up in 21b:
A point in $\mathbb{R}^2$ can also be represented as a **complex number** $z = x + iy \in \mathbb{C}$. The symbol i means now $\sqrt{-1}$. This is not only notational convenience. Complex numbers can be added and multiplied like other numbers and while $\mathbb{R}^2 = \mathbb{C}$, the later has a **multiplicative structure**. The basic rule to know is $i^2 = -1$ so that $(a+ib)(c+id) = ac - bd + i(ad+bc)$. An important observation of Euler is a link between the exponential and trigonometric functions:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

This implies for $\theta = \pi$ the **nicest formula** in mathematics [1]

$$\boxed{e^{i\pi} + 1 = 0}$$

It combines “calculus” in the form e, “geometry” in the form of π, “algebra” in the form of i, the additive unit 0 and the multiplicative unit 1.

[1] D. Wells, Which is the most beautiful?, Mathematical Intelligencer, 1988