Unit 6: Arc length

In this worksheet, we find the arc length of the **cycloid**

$$\vec{r}(t) = \begin{bmatrix} t - \sin(t) \\ \cos(t) \end{bmatrix}$$

from 0 to 2π. The curve is the solution to the famous **Brachistochrone problem**, the curve along which a ball descents fastest.

1 Compute the velocity $\vec{r}\,'(t)$.

2 Verify that $|\vec{r}\,'(t)| = \sqrt{2 - 2\cos t}$.

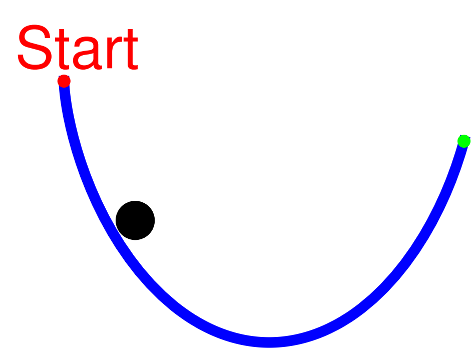
3 Use the double angle formula identity

$$2 - 2\cos(t) = 4\sin^2(\frac{t}{2}) \ .$$

to find the arc length

$$\int_0^{2\pi} |\vec{r}\,'(t)| \, dt \ .$$

Johann Bernoulli asked the Brachistochrone problem in 1696. The problem marks the start of a mathematical area called the **calculus of variations** in which one extremizes functions on infinite dimensional spaces.
Cycloids are curves traced by your feet, when you bike. It is a natural curve because it combines linear and circular motion.



About integration techniques

When looking at arc length integrals, basic integration techniques come back. Can you solve the following problems?

1 $\int_0^{2\pi} x\sin(x)\,dx$

2 $\int_0^{\sqrt{\pi}} 2x\sin(x^2)\,dx$.

3 $\int_0^1 \frac{1}{1+x^2}\,dx$.

4 $\int_0^{2\pi} \cos^2(x)\,dx$

1

[1] 1) -2π, 2) 2, 3) $\pi/4$. 4) π (use double angle formula)