6: Arc Length and Curvature

If $t \in [a,b] \mapsto \vec{r}(t)$ is a curve with velocity $\vec{r}\,'(t)$ and speed $|\vec{r}\,'(t)|$, then $L = \int_a^b |\vec{r}\,'(t)|\,dt$ is called the **arc length of the curve**. Written out in three dimensions, this is $L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt$.

1 The arc length of the **circle** of radius R given by $\vec{r}(t) = [R\cos(t), R\sin(t)]$ parameterized by $0 \leq t \leq 2\pi$ is 2π because the speed $|\vec{r}'(t)|$ is constant R. The answer is $2\pi R$.

2 The **helix** $\vec{r}(t) = [\cos(t), \sin(t), t]$ has velocity $\vec{r}\,'(t) = [-\sin(t), \cos(t), 1]$ and constant speed $|\vec{r}\,'(t)| = [-\sin(t), \cos(t), 1] = \sqrt{2}$.

3 What is the arc length of the curve

$$\vec{r}(t) = [t, \log(t), t^2/2]$$

for $1 \leq t \leq 2$? **Answer:** Because $\vec{r}\,'(t) = [1, 1/t, t]$, we have $\vec{r}\,'(t) = \sqrt{1 + \frac{1}{t^2} + t^2} = |\frac{1}{t} + t|$ and $L = \int_1^2 \frac{1}{t} + t\,dt = \log(t) + \frac{t^2}{2}|_1^2 = \log(2) + 2 - 1/2$.

4 Find the arc length of the curve $\vec{r}(t) = [3t^2, 6t, t^3]$ from $t = 1$ to $t = 3$.

5 What is the arc length of the curve $\vec{r}(t) = [\cos^3(t), \sin^3(t)], 0 \leq t \leq 2\pi$? **Answer:** We have $|\vec{r}'(t)| = 3\sqrt{\sin^2(t)\cos^4(t) + \cos^2(t)\sin^4(t)} = (3/2)|\sin(2t)|$. The absolute value forces us to split the integral into 4 intervals. Since $\int_0^{\pi/2} \sin(2t)\,dt = 1$, we have $\int_0^{2\pi} (3/2)|\sin(2t)|\,dt = (3/2)4 = 6$.

6 Find the arc length of $\vec{r}(t) = [t^2/2, t^3/3]$ for $-1 \leq t \leq 1$. This cubic curve satisfies $y^2 = x^3 8/9$ and is an example of an **elliptic curve**. The speed is $|\vec{r}(t)| = \sqrt{t^2 + t^4}$. Because $\int x\sqrt{1 + x^2}\,dx = (1 + x^2)^{3/2}/3$, the arc length integral can be evaluated using substitution by as $\int_{-1}^1 |t|\sqrt{1 + t^2}\,dx = 2\int_0^1 t\sqrt{1 + t^2}\,dt = 2(1 + t^2)^{3/2}/3|_0^1 = 2(2\sqrt{2} - 1)/3$.

7 The arc length of an **epicycle** $\vec{r}(t) = [t + \sin(t), \cos(t)]$ parameterized by $0 \leq t \leq 2\pi$. We have $|\vec{r}\,'(t)| = \sqrt{2 + 2\cos(t)}$. so that $L = \int_0^{2\pi} \sqrt{2 + 2\cos(t)}\,dt$. A **substitution** $t = 2u$ gives $L = \int_0^\pi \sqrt{2 + 2\cos(2u)}\;2du = \int_0^\pi \sqrt{2 + 2\cos^2(u) - 2\sin^2(u)}\;2du = \int_0^\pi \sqrt{4\cos^2(u)}\;2du = 4\int_0^\pi |\cos(u)|\,du = 8$.

8 Compute the arc length of the **catenary** $\vec{r}(t) = [t, e^t + e^{-t}]$ on an interval $[a,b]$ can be computed as $e^b - e^a - e^{-b} + e^{-a}$. By the way, $(e^t + e^{-t})/2$ is called the hyperbolic cosine and denoted by $\cosh(t)$.

Because a parameter change $t = t(s)$ corresponds to a **substitution** in the integration which does not change the integral, we immediately have

The arc length is independent of the parameterization of the curve.

9 The circle parameterized by $\vec{r}(t) = [\cos(t^2), \sin(t^2)]$ on $t = [0, \sqrt{2\pi}]$ has the velocity $\vec{r}\,'(t) = 2t(-\sin(t), \cos(t))$ and speed $2t$. The arc length is still $\int_0^{\sqrt{2\pi}} 2t\, dt = t^2|_0^{\sqrt{2\pi}} = 2\pi$.

10 We do not always have a closed formula for the arc length of a curve. The length of the **Lissajous figure** $\vec{r}(t) = [\cos(3t), \sin(5t)]$ leads to $\int_0^{2\pi} \sqrt{9\sin^2(3t) + 25\cos^2(5t)}\, dt$ which needs to be evaluated numerically.

Define the **unit tangent vector** $\vec{T}(t) = \vec{r}\,'(t)|/|\vec{r}\,'(t)|$ **unit tangent vector**.

The **curvature** if a curve at the point $\vec{r}(t)$ is defined as $\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$.

The curvature is the magnitude of the acceleration vector if $\vec{r}(t)$ traces the curve with constant speed 1. A large curvature at a point means that the curve turns sharply. Unlike the acceleration or the velocity, the curvature does not depend on the parameterization of the curve. You "see" the curvature, while you "feel" the acceleration.

The curvature does not depend on the parametrization.

Proof. If $s(t)$ be an other parametrization, then by the chain rule $d/dtT'(s(t)) = T'(s(t))s'(t)$ and $d/dtr(s(t)) = r'(s(t))s'(t)$. We see that the s' cancels in T'/r'.
Especially, if the curve is parametrized by arc length, meaning that the velocity vector $r'(t)$ has length 1, then $\kappa(t) = |T'(t)|$. It measures the rate of change of the unit tangent vector.

11 The curve $\vec{r}(t) = [t, f(t)]$, which is the graph of a function f has the velocity $\vec{r}\,'(t) = (1, f'(t))$ and the unit tangent vector $\vec{T}(t) = (1, f'(t))/\sqrt{1 + f'(t)^2}$. After some simplification we get

$$\kappa(t) = |\vec{T}'(t)|/|\vec{r}\,'(t)| = |f''(t)|/\sqrt{1 + f'(t)^2}^3$$

For example, for $f(t) = \sin(t)$, then $\kappa(t) = |\sin(t)|/|\sqrt{1 + \cos^2(t)}^3$.

If $\vec{r}(t)$ is a curve which has nonzero speed at t, then we can define $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$, the **unit tangent vector**, $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$, the **normal vector** and $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ the **bi-normal vector**. The plane spanned by $\vec{N}$ and $\vec{B}$ is called the **normal plane**. It is perpendicular to the curve. The plane spanned by T and N is called the **osculating plane**.

If we differentiate $\vec{T}(t) \cdot \vec{T}(t) = 1$, we get $\vec{T}'(t) \cdot \vec{T}(t) = 0$ and see that $\vec{N}(t)$ is perpendicular to $\vec{T}(t)$. Because B is automatically normal to T and N, we have shown:

The three vectors $(\vec{T}(t), \vec{N}(t), \vec{B}(t))$ are unit vectors orthogonal to each other.

A useful formula for curvature is

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

We prove this in class.