3: Lines and Planes

A point $P = (x_0, y_0, z_0)$ and a vector $\vec{v} = [a, b, c]$ define the **line**

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix} .$$

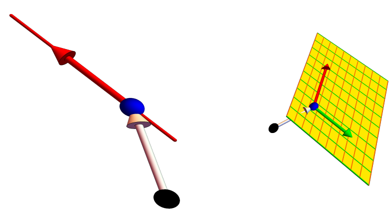
It is called the **parameterization** of the line.

Every vector contained inside the line is parallel to $\vec{v}$. We think about the parameter t as "time" and about $\vec{v}$ as the **velocity**. For $t = 0$, we are at P identified with $\vec{OP}$.

Given two points like $(2, 3, 4)$ with $(3, 3, 5)$, the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

connects the points $(2, 3, 4)$ with $(3, 3, 5)$. If t is restricted to $[0, 1]$ we get the **line segment** defined by the two points.



1 **Problem.** Parametrize the line through $P = (1, 1, 2)$ and $Q = (2, 4, 6)$.
Solution. with $\vec{v} = \vec{PQ} = [1, 3, 4]$ we get get the line

$$[x, y, z] = [1, 1, 2] + t[1, 3, 4]$$

which is $\vec{r}(t) = [1 + t, 1 + 3t, 2 + 4t]$. Since $[x, y, z] = [1, 1, 2] + t[1, 3, 4]$ consists of three equations $x = 1 + 2t, y = 1 + 3t, z = 2 + 4t$ we can solve each for t to get $t = (x - 1)/2 = (y - 1)/3 = (z - 2)/4$.

The line $\vec{r} = \vec{OP} + t\vec{v}$ defined by $P = (p, q, r)$ and vector $\vec{v} = [a, b, c]$ with nonzero a, b, c satisfies the **symmetric equations**

$$\frac{x - p}{a} = \frac{y - q}{b} = \frac{z - r}{c} .$$

Proof. Each of these expressions is equal to t. These symmetric equations have to be modified a bit if one or two of the numbers a, b, c are zero. If $a = 0$, replace the first equation with $x = p$, if $b = 0$ replace the second equation with $y = q$ and if $c = 0$ replace third equation with $z = r$.

2 Find the symmetric equations for the line through the two points $P = (0, 1, 1)$ and $Q = (2, 3, 4)$ **Solution.** first first form the parametric equations $[x, y, z] = [0, 1, 1] + t[2, 2, 3]$ or $x = 2t, y = 1 + 2t, z = 1 + 3t$ and solve for t to get $x/2 = (y - 1)/2 = (z - 1)/3$.

3 **Problem:** Find the symmetric equation for the z axes. **Answer:** This is a situation where $a = b = 0$ and $c = 1$. The symmetric equations are simply $x = 0, y = 0$. If two of the numbers a, b, c are zero, we have a coordinate plane. If one of the numbers are zero, then the line is contained in a coordinate plane.

A point P and two vectors $\vec{v}, \vec{w}$ define a **plane** $\vec{r}(t) = \vec{OP} + t\vec{v} + s\vec{w}$, where t, s are real numbers.
This is called the **parametric description** of a plane.

A point $P = (x_0, y_0, z_0)$ and two vectors $\vec{v}, \vec{w}$ defines the **plane**

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + s \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} .$$

It is called the **parameterization** of the plane.

2

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} .$$

If a plane contains the two vectors $\vec{v}$ and $\vec{w}$, then the vector

$$\vec{n} = \vec{v} \times \vec{w}$$

is orthogonal to both $\vec{v}$ and $\vec{w}$. Because also the vector $\vec{PQ} = \vec{OQ} - \vec{OP}$ is perpendicular to $\vec{n}$, we have $(Q - P) \cdot \vec{n} = 0$. With $Q = (x_0, y_0, z_0)$, $P = (x, y, z)$, and $\vec{n} = [a, b, c]$, this means $ax + by + cz = ax_0by_0 + cz_0 = d$. The plane is therefore described by a single equation $ax + by + cz = d$, where d is a constant obtained by plugging in a point. We have just shown

The equation of the plane $\vec{x} = \vec{x}_0 + t\vec{v} + s\vec{w}$

$$ax + by + cz = d ,$$

where $[a, b, c] = \vec{v} \times \vec{w}$ and d is obtained by plugging in $\vec{x}_0$.

3 **Problem:** Find the equation of a plane which contains the three points $P = (-1, -1, 1), Q = (0, 1, 1), R = (1, 1, 3)$.
Answer: The plane contains the two vectors $\vec{v} = [1, 2, 0]$ and $\vec{w} = [2, 2, 2]$. We have $\vec{n} = [4, -2, -2]$ and the equation is $4x - 2y - 2z = d$. The constant d is obtained by plugging in the coordinates of a point to the left. In our case, it is $4x - 2y - 2z = -4$.

4 **Problem:** Find the angle between the planes $x + y = -1$ and $x + y + z = 2$. **Answer:** find the angle between $\vec{n} = [1, 1, 0]$ and $\vec{m} = [1, 1, 1]$. It is $\arccos(2/\sqrt{6})$.