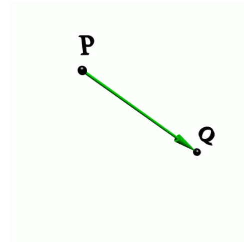
Unit 3: Distances

DISTANCE POINT-POINT. The distance between P and Q is

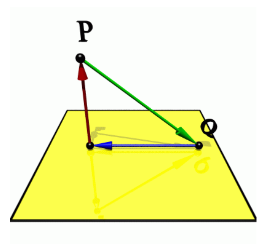
$$d(P,Q) = |\vec{PQ}| \; .$$



DISTANCE POINT-PLANE. The distance between a point P and a plane Σ given by $ax + by + cz = \vec{n} \cdot \vec{x} = d$ containing Q is

$$d(P,\Sigma) = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}$$

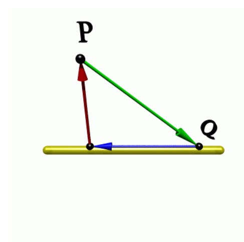
See it as the scalar projection of $\vec{PQ}$ onto $\vec{n} = [a,b,c]$. If the plane is parametrized $\vec{r} = \vec{OQ} + t\vec{v} + s\vec{w}$, then, with $\vec{n} = \vec{v} \times \vec{w}$, we can use that "height=volume/base area" holds in a parallelepiped.



DISTANCE POINT-LINE . If P is a point in space and L is the line $\vec{r}(t) = 0\vec{Q} + t\vec{u}$, then

$$d(P,L) = \frac{|\vec{PQ} \times \vec{u}|}{|\vec{u}|}$$

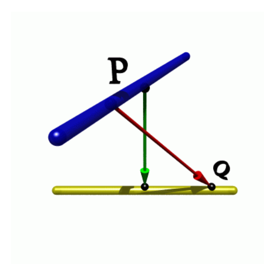
is the distance between P and the line L. Proof: the area divided by base length is the height of the parallelogram spanned by $\vec{PQ}$ and $\vec{u}$.



DISTANCE LINE-LINE . L is the line $\vec{r}(t) = Q + t\vec{u}$ and M is the line $\vec{s}(t) = P + t\vec{v}$, then

$$d(L,M) = \frac{|\vec{PQ} \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$$

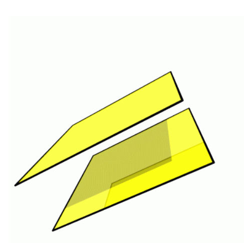
is the distance between the two lines L and M. Proof: the distance is the length of the vector projection of $\vec{PQ}$ onto $\vec{u} \times \vec{v}$.



DISTANCE PLANE-PLANE . If $\vec{n} \cdot \vec{x} = d$ and $\vec{n} \cdot \vec{x} = e$ are two parallel planes, then their distance is

$$\frac{|e-d|}{|\vec{n}|} \; .$$

Proof. If $\vec{x}$ satisfies $\vec{n} \cdot \vec{x} = d$, then $\vec{y} = \vec{x} + \vec{u}$ satisfies $\vec{n} \cdot \vec{y} = e$, where $\vec{u} = (e-d)\vec{n}/|\vec{n}|^2$ is perpendicular to the planes and has length $(e-d)/|\vec{n}|$. Non-parallel planes have distance 0.

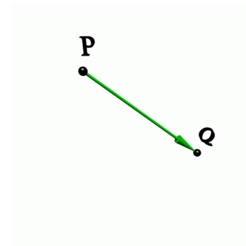


EXAMPLES

DISTANCE POINT-POINT . $P = (-5, 2, 4)$ and $Q = (-2, 2, 0)$ are two points, then

$$d(P, Q) = |\vec{PQ}| = \sqrt{(-5+2)^2 + (2-2)^2 + (0-4)^2} = 5 .$$

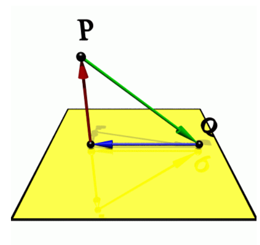
A question: what is the distance between the point $(-5, 2, 4)$ and the sphere $(x+2)^2 + (y-2)^2 + z^2 = 1$?



DISTANCE POINT-PLANE . $P = (7, 1, 4)$ is a point and $\Sigma : 2x + 4y + 5z = 9$ is a plane which contains the point $Q = (0, 1, 1)$. Then

$$d(P, \Sigma) = \frac{|[-7, 0, -3] \cdot [2, 4, 5]|}{|[2, 4, 5]|} = \frac{29}{\sqrt{45}}$$

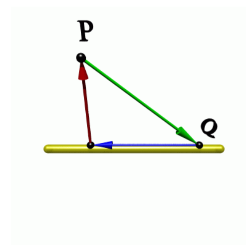
is the distance between P and Σ. Something to think about: without the absolute value, the result could become negative. What does this tell about the point P?



DISTANCE POINT-LINE . $P = (2, 3, 1)$ is a point in space and L is the line $\vec{r}(t) = [1, 1, 2] + t[5, 0, 1]$. Then

$$d(P, L) = \frac{|[-1, -2, 1] \times [5, 0, 1]|}{[5, 0, 1]} = \frac{|[-2, 6, 10]|}{\sqrt{26}} = \frac{\sqrt{140}}{\sqrt{26}}$$

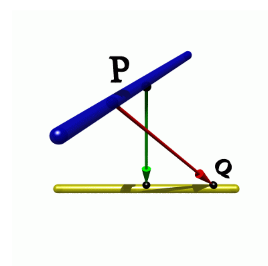
is the distance between P and L. Something to think about: what is the equation of the plane which contains the point P and the line L?



DISTANCE LINE-LINE . L is the line $\vec{r}(t) = [2, 1, 4] + t[-1, 1, 0]$ and M is the line $\vec{s}(t) = [-1, 0, 2] + t[5, 1, 2]$. The cross product of $[-1, 1, 0]$ and $[5, 1, 2]$ is $[2, 2, -6]$. The distance between these two lines is

$$d(L, M) = \frac{|[3, 1, 2] \cdot [2, 2, -6]|}{|[2, 2, -6]|} = \frac{4}{\sqrt{44}} .$$

Something to think about: also here, without the absolute value, the formula can give a negative result. What does this mean geometrically?



DISTANCE PLANE-PLANE . $5x+4y+3z = 8$ and $5x+4y+3z = 1$ are two parallel planes. Their distance is

$$\frac{|8-1|}{|[5, 4, 3]|} = \frac{7}{\sqrt{50}} .$$

Something to think about: what is the distance between the planes $x + 3y - 2z = 2$ and $5x + 15y - 10z = 30$?