3: Cross product

The **cross product** of two vectors $\vec{v} = [v_1, v_2, v_3]$ and $\vec{w} = [w_1, w_2, w_3]$ in space is defined as the vector

$$\vec{v} \times \vec{w} = [v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1] \, .$$

To remember it, we write the product as a "determinant":

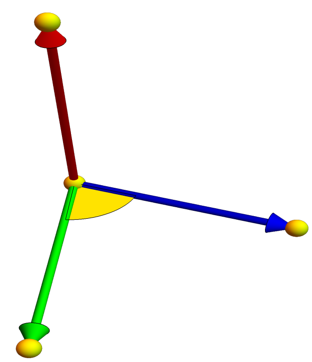
$$\begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = \begin{bmatrix} i & & \\ & v_2 & v_3 \\ & w_2 & w_3 \end{bmatrix} - \begin{bmatrix} & j & \\ v_1 & & v_3 \\ w_1 & & w_3 \end{bmatrix} + \begin{bmatrix} & & k \\ v_1 & v_2 & \\ w_1 & w_2 & \end{bmatrix}$$

which is $\vec{i}(v_2 w_3 - v_3 w_2) - \vec{j}(v_1 w_3 - v_3 w_1) + \vec{k}(v_1 w_2 - v_2 w_1)$. [1]

1 The cross product of $[1, 2, 3]$ and $[4, 5, 1]$ is the vector $[-13, 11, -3]$.

The cross product $\vec{v} \times \vec{w}$ is anti-commutative. The resulting vector is orthogonal to both $\vec{v}$ and $\vec{w}$.

Proof. We verify for example that $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$ and look at the definition.



The **sin** formula: $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}| \sin(\alpha)$.

Proof: We verify the **Lagrange's identity** $|\vec{v} \times \vec{w}|^2 = |\vec{v}|^2 |\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2$ by direct computation. Now, $|\vec{v} \cdot \vec{w}| = |\vec{v}||\vec{w}| \cos(\alpha)$.

The absolute value respectively length $|\vec{v} \times \vec{w}|$ defines the **area of the parallelogram** spanned by $\vec{v}$ and $\vec{w}$.

$\vec{v} \times \vec{w}$ is zero exactly if $\vec{v}$ and $\vec{w}$ are **parallel**, that is if $\vec{v} = \lambda \vec{w}$ for some real λ.

Proof. This follows immediately from the sin formula and the fact that $\sin(\alpha) = 0$ if $\alpha = 0$ or $\alpha = \pi$.

The cross product can therefore be used to check whether two vectors are parallel or not. Note that v and $-v$ are also considered parallel even so sometimes one calls this anti-parallel.

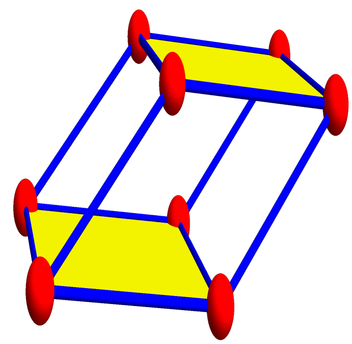
[1] It was Hamilton who found in 1843 a multiplication $*$ of 4 vectors. It contains both dot and cross product because $(0, v_1, v_2, v_3) * (0, w_1, w_2, w_3) = (-vw, v \times w)$.

The **trigonometric sin-formula**: if a, b, c are the side lengths of a triangle and α, β, γ are the angles opposite to a, b, c then $a/\sin(\alpha) = b/\sin(\beta) = c/\sin(\gamma$.

Proof. Twice the area of the triangle is $ab\sin(\gamma) = bc\sin(\alpha) = ac\sin(\beta)$ Divide the first equation by $\sin(\gamma)\sin(\alpha)$ to get one identity. Divide the second equation by $\sin(\alpha)\sin(\beta)$ to get the second identity.

2 If $\vec{v} = [a, 0, 0]$ and $\vec{w} = [b\cos(\alpha), b\sin(\alpha), 0]$, then $\vec{v} \times \vec{w} = [0, 0, ab\sin(\alpha)]$ which has length $|ab\sin(\alpha)|$.

The scalar $[\vec{u}, \vec{v}, \vec{w}] = \vec{u} \cdot (\vec{v} \times \vec{w})$ is called the **triple scalar product** of $\vec{u}, \vec{v}, \vec{w}$. The number $|[\vec{u}, \vec{v}, \vec{w}]|$ defines the **volume of the parallelepiped** spanned by $\vec{u}, \vec{v}, \vec{w}$ and the **orientation** of three vectors is the sign of $[\vec{u}, \vec{v}, \vec{w}]$.



The value $h = |\vec{u} \cdot \vec{n}|/|\vec{n}|$ is the height of the parallelepiped if $\vec{n} = (\vec{v} \times \vec{w})$ is a normal vector to the ground parallelogram of area $A = |\vec{n}| = |\vec{v} \times \vec{w}|$. The volume of the parallelepiped is $hA = (\vec{u} \cdot \vec{n}/|\vec{n}|)|\vec{v} \times \vec{w}|$ which simplifies to $\vec{u} \cdot \vec{n} = |(\vec{u} \cdot (\vec{v} \times \vec{w})|$ which is indeed the absolute value of the triple scalar product. The vectors $\vec{v}, \vec{w}$ and $\vec{v} \times \vec{w}$ form a **right handed coordinate system**. If the first vector $\vec{v}$ is your thumb, the second vector $\vec{w}$ is the pointing finger then $\vec{v} \times \vec{w}$ is the third middle finger of the right hand.

3 **Problem:** Find the volume of a **cuboid** of width a length b and height c. **Answer**. The cuboid is a parallelepiped spanned by $[a, 0, 0]$ $[0, b, 0]$ and $[0, 0, c]$. The triple scalar product is abc.

4 **Problem** Find the volume of the parallelepiped which has the vertices $O = (1, 1, 0), P = (2, 3, 1), Q = (4, 3, 1), R = (1, 4, 1)$. **Answer**: We first see that it is spanned by the vectors $\vec{u} = [1, 2, 1]$, $\vec{v} = [3, 2, 1]$, and $\vec{w} = [0, 3, 1]$. We get $\vec{v} \times \vec{w} = [-1, -3, 9]$ and $\vec{u} \cdot (\vec{v} \times \vec{w}) = 2$. The volume is 2.

5 **Problem:** find the equation $ax + by + cz = d$ for the plane which contains the point $P = (1, 2, 3)$ as well as the line which passes through $Q = (3, 4, 4)$ and $R = (1, 1, 2)$. To do so find a vector $\vec{n} = [a, b, c]$ normal to the and noting $(\vec{x} - \vec{OP}) \cdot \vec{n} = 0$. **Answer:** A normal vector $\vec{n} = [1, -2, 2] = [a, b, c]$ of the plane $ax + by + cz = d$ is obtained as the cross product of $\vec{PQ}$ and $\vec{RQ}$ With $d = \vec{n} \cdot \vec{OP} = [1, -2, 2] \cdot [1, 2, 3] = 3$, we get the equation $x - 2y + 2z = 3$.

The cross product appears in physics, like for the angular momentum, the Lorentz force or the Coriolis force. We will however mainly use the cross product for constructions like to get the equation of a plane through 3 points A, B, C.