2: Vectors and Dot product

Two points $P = (a,b,c)$ and $Q = (x,y,z)$ in space define a **vector** $\vec{PQ} = \vec{v} = [x-a, y-b-z-c]$ pointing from P to Q. The real numbers v_1, v_2, v_3 in $\vec{v} = [v_1, v_2, v_3]$ are the **components** of $\vec{v}$.

Similar definitions hold in two dimensions, where vectors have two components. Vectors can be drawn **everywhere** in space. Two vectors with the same components are considered **equal**. [1]

The **addition** of two vectors is $\vec{u}+\vec{v} = [u_1,u_2,u_3]+[v_1,v_2,v_3] = [u_1+v_1, u_2+v_2, u_3+v_3]$. The **scalar multiple** $\lambda\vec{u} = \lambda[u_1,u_2,u_3] = [\lambda u_1, \lambda u_2, \lambda u_3]$. The difference $\vec{u}-\vec{v}$ can be seen as the addition of $\vec{u}$ and $(-1)\cdot\vec{v}$.

The addition and scalar multiplication of vectors satisfy the laws you know from **arithmetic**. **commutativity** $\vec{u}+\vec{v} = \vec{v}+\vec{u}$, **associativity** $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ and $r*(s*\vec{v}) = (r*s)*\vec{v}$ as well as **distributivity** $(r+s)\vec{v} = \vec{v}(r+s)$ and $r(\vec{v}+\vec{w}) = r\vec{v}+r\vec{w}$, where $*$ is scalar multiplication.

The **length** or **magnitude** $|\vec{v}|$ of a vector $\vec{v} = \vec{PQ}$ is defined as the distance $d(P,Q)$ from P to Q. A vector of length 1 is called a **unit vector**. A synonym is **direction**. Nonzero vectors have length and magnitude.

1 $|[3,4]| = 5$ and $|[3,4,12]| = 13$. Examples of unit vectors are $|\vec{i}| = |\vec{j}| = \vec{k}| = 1$ and $[3/5, 4/5]$ and $[3/13, 4/13, 12/13]$. The only vector of length 0 is the zero vector $|\vec{0}| = 0$.

The **dot product** of two vectors $\vec{v} = [a,b,c]$ and $\vec{w} = [p,q,r]$ is defined as $\vec{v}\cdot\vec{w} = ap+bq+cr$.

The dot product determines distance and distance determines the dot product.

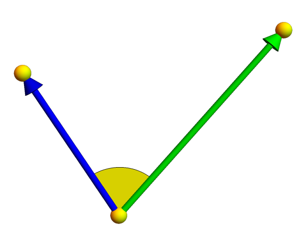
Proof: Using the dot product one can express the length of $\vec{v}$ as $|\vec{v}| = \sqrt{\vec{v}\cdot\vec{v}}$. On the other hand, $(\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w}) = \vec{v}\cdot\vec{v}+\vec{w}\cdot\vec{w}+2(\vec{v}\cdot\vec{w})$ allows to solve for $\vec{v}\cdot\vec{w}$:

$$\vec{v}\cdot\vec{w} = (|\vec{v}+\vec{w}|^2 - |\vec{v}|^2 - |\vec{w}|^2)/2 .$$

The **Cauchy-Schwarz inequality** tells $|\vec{v}\cdot\vec{w}| \leq |\vec{v}||\vec{w}|$.

Proof. We only need to show it in the case $|\vec{w}| = 1$. Define $a = \vec{v}\cdot\vec{w}$ and estimate $0 \leq (\vec{v}-a\vec{w})\cdot(\vec{v}-a\vec{w})$ to get $0 \leq (\vec{v}-(\vec{v}\cdot\vec{w})\vec{w})\cdot(\vec{v}-(\vec{v}\cdot\vec{w})\vec{w}) = |\vec{v}|^2+(\vec{v}\cdot\vec{w})^2-2(\vec{v}\cdot\vec{w})^2 = |\vec{v}|^2-(\vec{v}\cdot\vec{w})^2$ which means $(\vec{v}\cdot\vec{w})^2 \leq |\vec{v}|^2$.

The **angle** between two nonzero vectors is defined as the unique $\alpha \in [0,\pi]$ which satisfies $\vec{v}\cdot\vec{w} = |\vec{v}|\cdot|\vec{w}|\cos(\alpha)$.



[1] We cover 2400 years of math from Pythagoras (500 BC), Al Kashi (1400), Cauchy (1800) to Hamilton (1850).

Al Kashi's theorem: A triangle ABC with side lengths a, b, c and angle α opposite to c satisfies $a^2 + b^2 = c^2 + 2ab\cos(\alpha)$.

Proof. Define $\vec{v} = \vec{AB}, \vec{w} = \vec{AC}$. Because $c^2 = |\vec{v} - \vec{w}|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = |\vec{v}|^2 + |\vec{w}|^2 - 2\vec{v} \cdot \vec{w}$, We know $\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cos(\alpha)$ so that $c^2 = |\vec{v}|^2 + |\vec{w}|^2 - 2|\vec{v}| \cdot |\vec{w}| \cos(\alpha) = a^2 + b^2 - 2ab\cos(\alpha)$.

The **triangle inequality** tells $|\vec{u} + \vec{v}| \leq |\vec{u}| + |\vec{v}|$

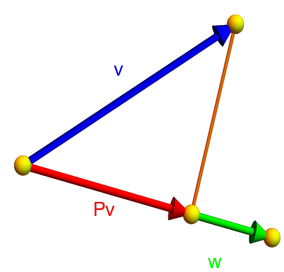
Proof: $|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) = \vec{u}^2+\vec{v}^2+2\vec{u}\cdot\vec{v} \leq \vec{u}^2+\vec{v}^2+2|\vec{u}\cdot\vec{v}| \leq \vec{u}^2+\vec{v}^2+2|\vec{u}|\cdot|\vec{v}| = (|\vec{u}|+|\vec{v}|)^2$.

Two vectors are called **orthogonal** or **perpendicular** if $\vec{v} \cdot \vec{w} = 0$. The zero vector $\vec{0}$ is orthogonal to any vector. For example, $\vec{v} = [2, 3]$ is orthogonal to $\vec{w} = [-3, 2]$.

Pythagoras theorem: if $\vec{v}$ and $\vec{w}$ are orthogonal, then $|\vec{v} - \vec{w}|^2 = |\vec{v}|^2 + |\vec{w}|^2$.

Proof: $(\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} + 2\vec{v} \cdot \vec{w} = \vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w}$. [2]

The vector $\mathrm{P}(\vec{v}) = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}$ is called the **projection** of $\vec{v}$ onto $\vec{w}$. The **scalar projection** $\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}$ is plus or minus the length of the projection of $\vec{v}$ onto $\vec{w}$. The vector $\vec{b} = \vec{v} - P(\vec{v})$ is a vector orthogonal to $\vec{w}$.



2 Find the projection of $\vec{v} = [0, -1, 1]$ onto $\vec{w} = [1, -1, 0]$. **Answer:** $\mathrm{P}(\vec{v}) = [1/2, -1/2, 0]$.

3 A wind force $\vec{F} = [2, 3, 1]$ is applied to a car which drives in the direction of the vector $\vec{w} = [1, 1, 0]$. Find the projection of $\vec{F}$ onto $\vec{w}$, the force which accelerates or slows down the car. **Answer**: $\vec{w}(\vec{F} \cdot \vec{w}/|\vec{w}|^2) = [5/2, 5/2, 0]$.

4 How can we visualize the dot product? **Answer**: the absolute value of the dot product is the length of the projection. Positive dot product means $\vec{v}$ and $\vec{w}$ form an acute angle, negative if that angle is obtuse.

5 Given $\vec{v} = [2, 1, 2]$ and $\vec{w} = [3, 4, 0]$. Find a vector which is in the plane defined by $\vec{v}$ and $\vec{w}$ and which bisects the angle between these two vectors. **Answer**. Normalize the two vectors to make them unit vectors then add them to get $[13, 17, 10]/15$.

6 Given two vectors $\vec{v}, \vec{w}$ which are perpendicular. Under which condition is $\vec{v}+\vec{w}$ perpendicular to $\vec{v} - \vec{w}$? **Answer**: Find the dot product of $\vec{v} + \vec{w}$ with $\vec{v} - \vec{w}$ and set it zero.

7 Is the angle between $[1, 2, 3]$ and $[-15, 2, 4]$ acute or obtuse? **Answer**: the dot product is 1. Ah! Cute!

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[2] We have just proven Pythagoras and Al Kashi. Distance and angle were defined, not deduced.