

## Lecture 2: Functions

A **function** is a rule which assigns to a real number a new real number. The function  $f(x) = x^2 - 2x$  for example assigns to the number  $x = 4$  the value  $4^2 - 8 = 8$ . A function is given with a **domain**  $A$ , the points where  $f$  is defined and a **codomain**  $B$  a set of numbers which  $f$  can reach. Usually, functions are defined everywhere, like the function  $f(x) = x^2 - 2x$ . Other functions like  $g(x) = 1/x$  can not be evaluated at 0.

We have some flexibility to define the domain and codomain. Let  $R_0$  be the set of all real numbers which do not contain 0. If we equip  $f(x) = 1/x$  with the domain and codomain  $R_0$  then  $f$  is a map from  $R_0$  to  $R_0$  and it is its own inverse. Here are a few examples of functions. We will look at them in more detail during the lecture. Very important are polynomials, trigonometric functions, the exponential and logarithmic function. You won't find the  $h$ -exponentials and  $h$ -logarithms in textbooks. But they will be important for us. They are the exponentials and logarithms in calculus without limit. We will see later that in the limit  $h \rightarrow 0$  they become the regular exponential and logarithm.

constant	$f(x) = 1$	power	$f(x) = 2^x$
identity	$f(x) = x$	exponential	$f(x) = e^x = \exp(x)$
linear	$f(x) = 3x + 1$	logarithm	$f(x) = \log(x) = \exp^{-1}(x)$
quadratic	$f(x) = x^2$	absolute value	$f(x) =  x $
cosine	$f(x) = \cos(x)$	devil comb	$f(x) = \sin(1/x)$
sine	$f(x) = \sin(x)$	bell function	$f(x) = e^{-x^2}$
h-exponentials	$f(x) = \exp_h(x) = (1+h)^{x/h}$	witch of Agnesi	$f(x) = \frac{1}{1+x^2}$
h-logarithms	$f(x) = \log_h(x) = \exp_h^{-1}$	sinc	$\sin(x)/x$

We can build new functions by:

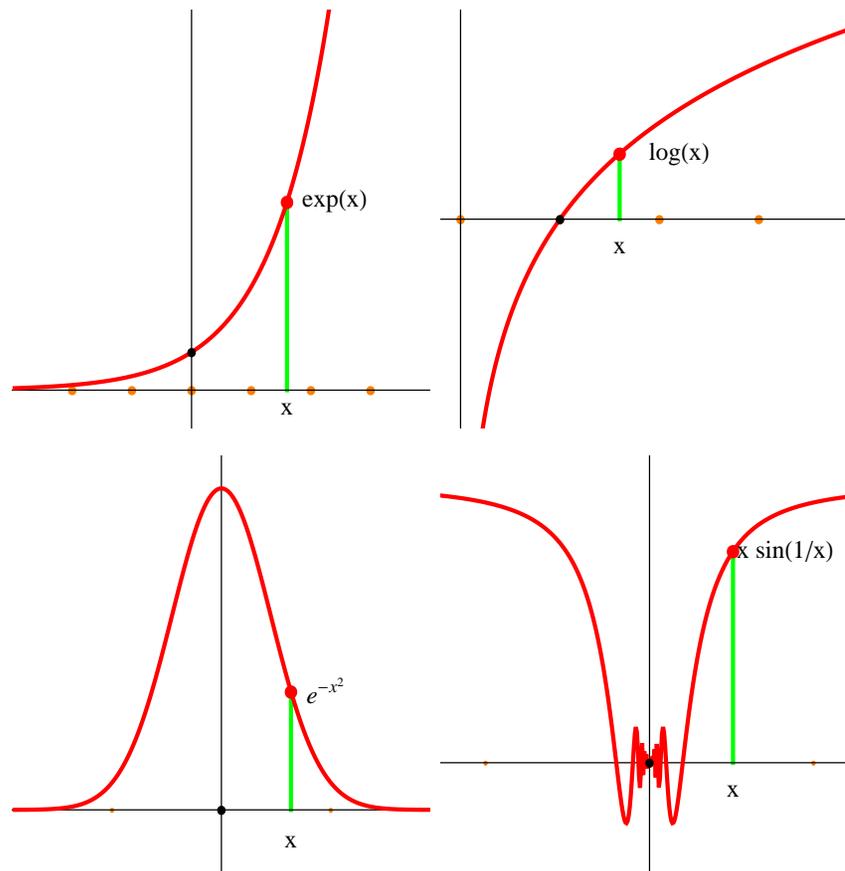
addition	$f(x) + g(x)$
scaling	$2f(x)$
translate	$f(x + 1)$
compose	$f(g(x))$
invert	$f^{-1}(x)$
difference	$f(x + 1) - f(x)$
sum up	$f(x) + f(x + 1) + \dots$

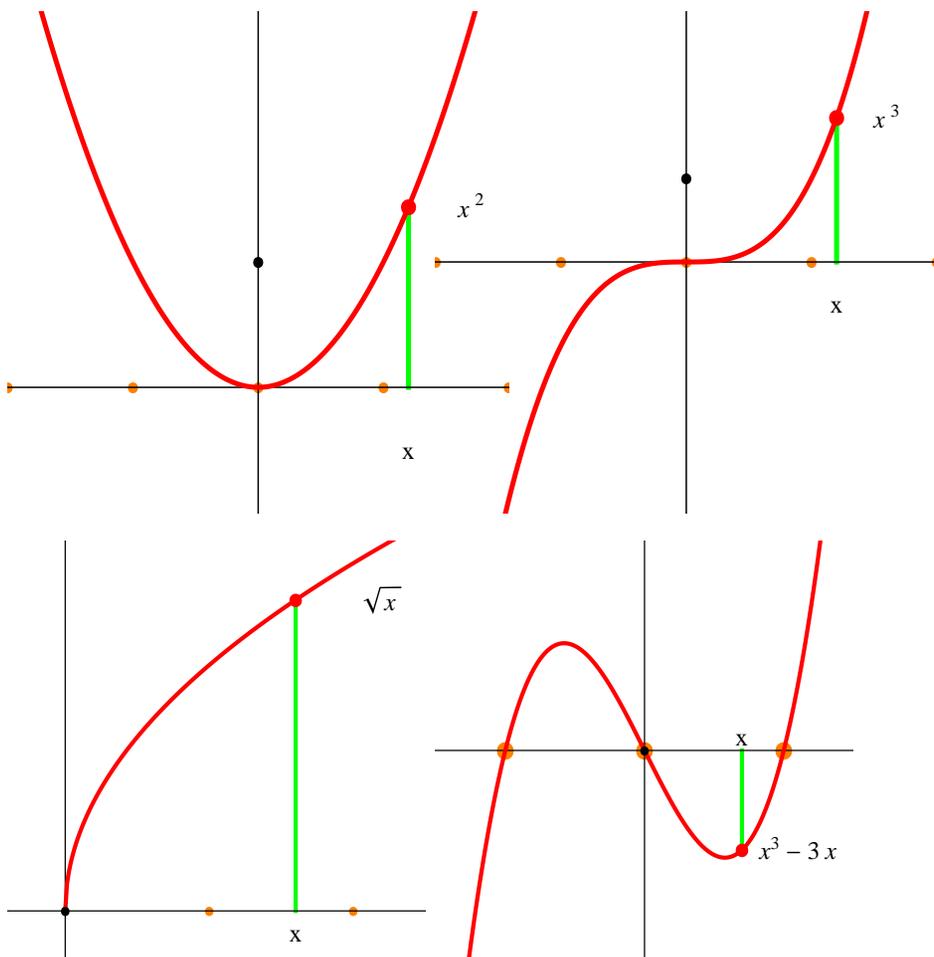
Here are important functions:

polynomials	$x^2 + 3x + 5$
rational functions	$(x + 1)/(x^4 + 1)$
exponential	$e^x$
logarithm	$\log(x)$
trig functions	$\sin(x), \tan(x)$
inverse trig functions	$\arcsin^{-1}(x), \arctan(x)$ .
roots	$\sqrt{x}, x^{1/3}$

We will look at these functions **a lot** during this course. The logarithm, exponential and trigonometric functions are especially important. For some functions, we need to restrict the domain, where the function is defined. For the square root function  $\sqrt{x}$  or the logarithm  $\log(x)$  for example, we have to assume that the number  $x$  on which we evaluate the function is positive. We write that the domain is  $(0, \infty) = \mathbf{R}^+$ . For the function  $f(x) = 1/x$ , we have to assume that  $x$  is different from zero. Keep these three examples in mind.

The **graph** of a function is the set of points  $\{(x, y) = (x, f(x))\}$  in the plane, where  $x$  runs over the domain  $A$  of  $f$ . Graphs allow us to **visualize** functions. We can "see them", when we draw the graph.





## Homework

- 1 Draw the function  $f(x) = x \cos(x)$ . Its graph goes through the origin  $(0, 0)$ . You can use Wolfram Alpha or a calculator if you like.

  - a) A function is called **odd** if  $f(-x) = -f(x)$ . Is  $f$  odd?
  - b) A function is called **even** if  $f(x) = f(-x)$ . Is  $f$  even?
  - c) A function is called **monotone increasing** if  $f(y) > f(x)$  if  $y > x$ . Is  $f$  monotone increasing on the interval  $[-1, 1]$ ? There is need to decide this yet analytically. Just draw<sup>(\*)</sup> and decide.
- 2 A function  $f : A \rightarrow B$  is called **invertible** or **one to one** if there is an other function  $g$  such that  $g(f(x)) = x$  for all  $x$  in  $A$  and  $f(g(y)) = y$  for all  $y \in B$ . In that case, the function  $g$  is called the **inverse** of  $f$ . For example, the function  $g(x) = \sqrt{x}$  is the inverse of  $f(x) = x^2$  as a function from  $A = [0, \infty)$  to  $B = [0, \infty)$ . Determine from the following functions whether they are invertible. If they are invertible, find the inverse.

  - a)  $f(x) = \sin(x)$  from  $A = [0, \pi/2]$  to  $B = [0, 1]$
  - b)  $f(x) = x^7$  from  $A = \mathbf{R}$  to  $B = \mathbf{R}$
  - c)  $f(x) = x^4$  from  $A = \mathbf{R}$  to  $B = \mathbf{R}$
  - d)  $f(x) = \exp(x)$  from  $A = \mathbf{R}$  to  $B = \mathbf{R}^+ = (0, \infty)$ .
  - e)  $f(x) = 1/(1 + x^2)$  from  $A = [0, \infty)$  to  $B = [0, \infty)$ .
- 3 We have defined  $\log_h(x)$  and  $\exp_h(x)$  in the text.

  - a) Draw the graphs of  $\exp_1(x)$ ,  $\exp_{1/2}(x)$ ,  $\exp_{1/10}(x)$  and  $\exp(x)$ .
  - b) Draw the graphs of  $\log_1(x)$ ,  $\log_{1/2}(x)$ ,  $\log_{1/10}(x)$  and  $\log(x)$ .
- 4 Plot the Function  $\exp(\exp(\exp(x)))$ . The computer indicates that this functions blows up to infinity before  $x = 10$ . Is this true? Give a reason. Even Mathematica, a professional computer algebra system tells when evaluating that number "General::ovfl: Overflow occurred in computation.". Can you still estimate how many digits this number has? We make here a little competition and see who comes closest.
- 5 A function  $f(x)$  has a **root** at  $x = a$  if  $f(a) = 0$ . Roots are places, where the function is zero. Find one root for each of the following functions or state that there is none.

a) $f(x) = \cos(x)$	d) $f(x) = \log(x)$
b) $f(x) = \exp(-x^4)$	e) $f(x) = \sin(x) - 1$
c) $f(x) = x^5 - x$	f) $f(x) = \csc(x) = 1/\sin(x)$

(\*) Here is how you can use the Web to plot a function. The example given is  $\sin(x)$ .

[http://www.wolframalpha.com/input/?i=Plot+sin\(x\)](http://www.wolframalpha.com/input/?i=Plot+sin(x))