Lecture 6: Fundamental theorem

Calculus is the theory of **differentiation** and **integration**. We fix here a positive constant h and take differences and sums. Without taking limits, we prove a version of the fundamental theorem of calculus and differentiate and integrate polynomials, exponentials and trigonometric functions.

> Given a function, define the **differential quotient**
>
> $$Df(x) = (f(x+h) - f(x))\,\frac{1}{h}$$
>
> If f is continuous then Df is a continuous function too. We call it also "derivative".

1 Lets take the constant function $f(x) = 5$. We get $Df(x) = (f(x+h) - f(x))/h = (5-5)/h = 0$ everywhere. We see that in general if f is a constant function, then $Df(x) = 0$.

2 $f(x) = 3x$. We have $Df(x) = (f(x+h) - f(x))/h = (3(x+h) - 3x)/h$ which is $\boxed{3}$.

3 If $f(x) = ax + b$, then $Df(x) = \boxed{a}$.

For constant functions, the derivative is zero. For linear functions, it is the slope.

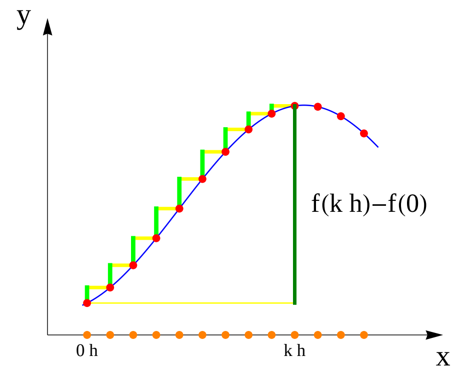
4 For $f(x) = x^2$ we compute $Df(x) = ((x+h)^2 - x^2)/h = (2hx + h^2)/h$ which is $\boxed{2x+h}$.

> Given a function f define a new function $Sf(x)$ by summing up all values of $f(hj)$ where $0 \leq jh < x$. That is, if k is such that kh is the largest below x, then
>
> $$Sf(x) = h[\ f(0) + f(h) + f(2h) + ... + f(kh)\ ]$$
>
> We call Sf also the "integral" or "antiderivative" of f.

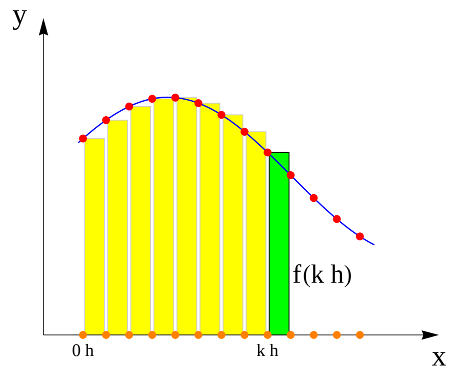
5 Compute $Sf(x)$ for $f(x) = 1$. **Solution**. We have $Sf(x) = 0$ for $x \leq h$, and $Sf(x) = h$ for $h \leq x < 2h$ and $Sf(x) = 2h$ for $2h \leq x < 3h$. In general $S1(jh) = j$ and $S1(x) = kh$ where k is the largest integer such that $kh < x$. The function g grows linearly but quantized steps.

The difference	$Df(x)$	will become the	**derivative**	$f'(x)$	.
The sum	$Sf(x)$	will become the	**integral**	$\int_0^x f(t)\ dt$	.

Df	means	rise over run	and is close to	the slope	of the	graph of f.
Sf	means	areas of rectangles	and is close to	the area	under the	graph of f.



Theorem: Sum the differences and get $SDf(kh) = f(kh) - f(0)$



Theorem: Difference the sum and get $DSf(kh) = f(kh)$

6 For $f(x) = [x]_h^m = x(x-h)(x-2h)...(x-mh+h)$ we have

$$f(x+h) - f(x) = (x(x-h)(x-2h)...(x-kh+2h))\,((x+h) - (x-mh+h)) = [x]^{m-1}hm$$

and so $D[x]_h^m = m[x]_h^{(m-1)}$. Lets leave the h away to get the important formula $\boxed{D[x]^m = m[x]^{m-1}}$

We can establish from this differentiation formulas for **polynomials**.

7 If $f(x) = [x] + [x]^3 + 3[x]^5$ then $Df(x) = 1 + 3[x]^2 + 15[x]^4$.
The fundamental theorem allows us to integrate and get the right values at the points k/n:

8 Find Sf for the same function. The answer is $Sf(x) = [x]^2/2 + [x]^4/4 + 3[x]^6/6$.

> Define $\exp_h(x) = (1+h)^{x/h}$. It is equal to 2^x for $h = 1$ and morphs into the function e^x when h goes to zero. As a rescaled exponential, it is continuous and monotone.

9 The function $\exp_h(x) = (1+h)^{x/h}$ satisfies $D\exp_h(x) = \exp_h(x)$.
Solution: $\exp_h(x+h) = (1+h)\exp(x)$ shows that. $\boxed{D\exp_h(x) = \exp_h(x)}$

10 Define $\exp^a(x) = (1+ah)^{x/h}$. Now $D\exp_h^a(x) = a\exp_h^a(x)$. Since $\exp_h(ax)$ is not equal to $\exp_h^a(x)$, we write also $e_h^{a \cdot x} = \exp_h(a \cdot x) = \exp_h^a(x)$. Now: $\boxed{D\exp_h(a \cdot x) = a\exp_h(a \cdot x)}$

11 We can also replace a with the complex ai and consider $\exp_h^{ai}(x) = (1+aih)^{x/h}$. Now, $D\exp_h^{ai}(x) = ai\exp_h^{ai}(x)$. Real and imaginary parts define new functions $\exp_h^{ai}(x) = \cos_h(a \cdot x) + i\sin_h(a \cdot x)$. We have $D\sin_h(a \cdot x) = a\cos_h(a \cdot x)$ and $D\cos_h(a \cdot x) = -a\sin_h(a \cdot x)$. These functions morph into the familiar cos and sin functions for $h \to 0$. But in general, for any h and any a, we have $\boxed{D\cos_h(a \cdot x) = -a\sin_h(a \cdot x) \text{ and } D\sin_h(a \cdot x) = a\cos_h(a \cdot x).}$

Homework

We leave the h away in this homework. To have more fun, also define $\log_h$ as the inverse of $\exp_h$ and define $1/[x]_h = D\log_h(x)$ for $x > 0$. If we start integrating from 1 instead of 0 as usual we have $S_1 1/[x]_h = \log_h(x)$. [1] We also write here x^n for $[x]_h^n$ and write $\exp(a \cdot x) = e^{a \cdot x}$ instead of $\exp_h^a(x)$ and $\log(x)$ instead of $\log_h(x)$ because we are among friends. Use the differentiation and integration rules on the right to find derivatives and integrals of the following functions:

1 Find the derivatives $Df(x)$ of the following functions:

a) $f(x) = x^2 + 6x^7 + x$
a) $f(x) = x^4 + \log(x)$
c) $f(x) = -3x^3 + 17x^2 - 5x$. What is $Df(0)$?

2 Find the integrals $Sf(x)$ of the following functions:

a) $f(x) = x^4$.
b) $f(x) = x^2 + 6x^7 + x$
c) $f(x) = -3x^3 + 17x^2 - 5x$. What is $Sf(1)$?

3 Find the derivatives $Df(x)$ of the following functions

a) $f(x) = \exp(3 \cdot x) + x^6$
b) $f(x) = 4\exp(-3 \cdot x) + 9x^6$
c) $f(x) = -\exp(5 \cdot x) + x^6$

4 Find the integrals $Sf(x)$ of the following functions

a) $f(x) = \exp(6 \cdot x) - 3x^6$
b) $f(x) = \exp(8 \cdot x) + x^6$
c) $f(x) = -\exp(5 \cdot x) + x^6$

5 Define $f(x) = \sin(4 \cdot x) - \exp(2 \cdot x) + x^4$ and assume $h = 1/100$ in part c).
a) Find $Df(x)$
b) Find $Sf(x)$
c) Determine the value of

$$\frac{1}{100}[f(\frac{0}{100}) + f(\frac{1}{100}) + \cdots + f(\frac{99}{100})] .$$

[1] We do not see h in daily lives, or do we? An allegory: in our universe, where $h = 1.616 \cdot 10^{-35} m$, the difference between the $\sin_h$ and $\sin$ is so small that a x-ray oscillating with $\nu = 10^{17}$ Herz traveling for 13 billion years $t = 4 \cdot 10^{17} s$ would only start to deviate noticeably from the classical $\sin(x)$ wave when it reaches us at $\nu \cdot t = 4 \cdot 10^{34}$ oscillations. Since $\sin_h(x) - \sin(x)$ only starts to grow at around $x = 1/h \sim 10^{35}$ oscillations, the x-ray would **look the same** when using the trig functions $\sin_h, \cos_h$. If γ is in the Gamma ray spectrum $10^{19} Hz$, the functions $\sin_h, \cos_h$ start to grow in amplitude earlier. A γ wave emitted 1 billion years ago would be observed as a **Gamma ray burst**.

All calculus on 1/3 page

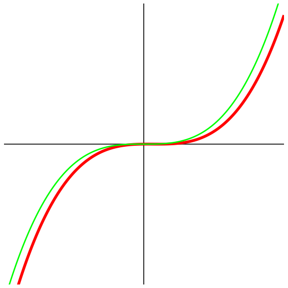
Fundamental theorem of Calculus: $DSf(x) = f(x)$ and $SDf(x) = f(x) - f(0)$.

Differentiation rules
$Dx^n = nx^{n-1}$
$De^{a \cdot x} = ae^{a \cdot x}$
$D\cos(a \cdot x) = -a\sin(a \cdot x)$
$D\sin(a \cdot x) = a\cos(a \cdot x)$
$D\log(x) = 1/x$

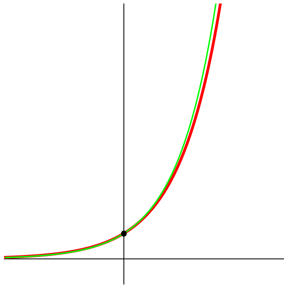
Integration rules (for $x = kh$)
$Sx^n = x^{n+1}/(n+1)$
$Se^{a \cdot x} = (e^{a \cdot x} - 1)/a$
$S\cos(a \cdot x) = \sin(a \cdot x)/a$
$S\sin(a \cdot x) = -\cos(a \cdot x)/a$
$S\frac{1}{x} = \log(x)$

Fermat's extreme value theorem: If $Df(x) = 0$ and f is continuous, then f has a local maximum or minimum in the open interval $(x, x+h)$.

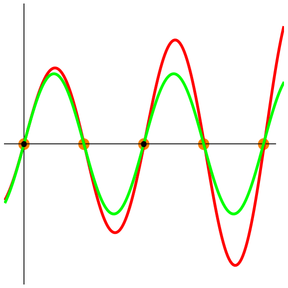
Pictures



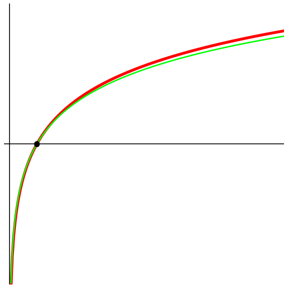
$[x]_h^3$ for $h = 0.1$



$\exp_h(x)$ for $h = 0.1$



$\sin_h(x)$ for $h = 0.1$



$\log_h(x)$ for $h = 0.1$

Here are some worked out examples, similar to what we expect you to do for the homework of lecture 6: The homework should be straightforward, except when finding $Sf(x)$, we want to add a constant such that $Sf(0) = 0$. In general, you will not need to evaluate functions and can leave terms like $\sin(5 \cdot x)$ as they are. If you have seen calculus already, then you could do this exercice by writing

$$\frac{d}{dx} f(x)$$

instead of $Df(x)$ and by writing

$$\int_0^x f(x)\, dx$$

instead of $Sf(x)$. Since we did not introduce the derivative df/dx nor the integral $\int_0^x$ yet, for now, just use the differentiation and integrations rules in the box to the right to solve the problems.

1 **Problem:** Find the derivative $Df(x)$ of the function $f(x) = \sin(5 \cdot x) + x^7 + 3$.
Answer: From the differentiation rules, we know $\boxed{Df(x) = 5\cos(5 \cdot x) + 7x^6}$.

2 **Problem:** Find the derivative $Df(0)$ of the same function $f(x) = \sin(5 \cdot x) + 5x^7 + 3$.
Answer: We know $Df(x) = 5\cos(5 \cdot x) + 35x^6$. Plugging in $x = 0$ gives $\boxed{5}$.

3 **Problem:** Find the integral $Sf(x)$ of the function $f(x) = \sin(5 \cdot x) + 5x^7 + 3$.
Answer: From the integration rules, we know $\boxed{Sf(x) = -\cos(5 \cdot x)/5 + 5x^8/8 + 3x}$.

4 **Problem:** Find the integral $Sf(1)$ of the function $f(x) = x^2 + 1$.
Answer: From the integration rules, we know $Sf(x) = x^3/3 + x$. Plugging in $x = 1$ gives $\boxed{1/3 + 1}$ if we use the functions in the limit $h \to 0$. For positive h, we have to evaluate $x(x-h)(x-2h)/3 + x$ for $x = 1$ which is $(1-h)(1-2h)/3 + 1$

5 **Problem:** Find the integral $Sf(1)$ of the function $f(x) = \exp(4 \cdot x)$.
Answer: From the integration rules, we know $Sf(x) = \exp(4 \cdot x)/4 - 1/4$. We have added a constant such that $Sf(0) = 0$. Plugging in $x = 1$ gives $\boxed{\exp(4)/4 - 1/4}$.

6 **Problem:** Assume $h = 1/1000$. Determine the value of

$$\frac{1}{1000}[f(\frac{0}{1000}) + f(\frac{1}{1000}) + ... + f(\frac{999}{1000})]$$

for the function $f(x) = -\sin(7x) + \exp(3x)$.
Answer: The problem asks for $Sf(1)$. We first compute $Sf(x)$ taking care that $Sf(0) = 0$.

$$Sf(x) = \cos(7x)/7 + \exp(3x)/3 - (1/7 + 1/3)\ .$$

Now plug in $x = 1$ to get $\boxed{\cos(7)/7 + \exp(3)/3 - (1/7 + 1/3)}$.