Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2024

2/28/2024: First Hourly

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

Please write neatly. Use the same page for the answer if possible.

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		100

Problem 1) TF questions (10 points) No justifications are needed.

1) [T] [F] $\lim_{x\to 0} \frac{\sin(x)}{x} = 0$.

Solution:
The limit is 1.

2) [T] [F] e^x has a root at $x = 0$.

Solution:
There is no root

3) [T] [F] $\lim_{x\to 0} \cos(x) = 1$.

Solution:
You know that $\cos(0) = 1$.

4) [T] [F] $f(x) = |\sin(x)|$ is differentiable everywhere.

Solution:
There are points where the function is only continuous

5) [T] [F] $f(x) = \frac{1-x^5}{1-x}$ has a limit at $x = 1$.

Solution:
Divide through

6) [T] [F] The function $f(x) = x^7 + 7$ has a critical point at $x = 0$.

Solution:
Yes, even so it is neither a max nor min.

7) [T] [F] The function $f(x) = \sin(x)\sin(\frac{1}{x})$ (with the understanding $f(0) = 0$) is continuous everywhere.

Solution:
By the squeeze theorem.

8) [T] [F] $\frac{d}{dx}\ln|5x| = \frac{5}{x}$.

Solution:
An important one. The derivative is $5/(5x) = 1/x$.

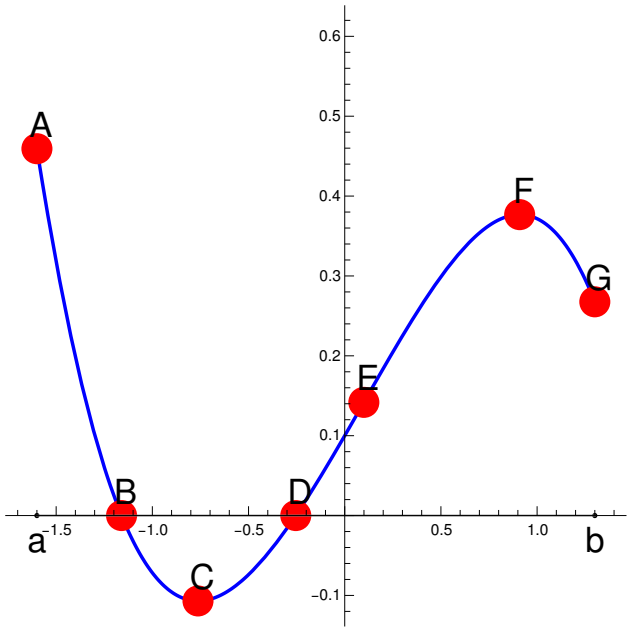
9) [T] [F] $\frac{d}{dx}\ln|7-2x| = \frac{-2}{7-2x}$.

Solution:
This is correct

10) [T] [F] The derivative of $\frac{g}{f}$ is $\frac{fg'-f'g}{f^2}$.

Solution:
Yes, even so we have changed the letters

Problem 2) Analysis (10 points) No justifications are needed.



a) (2 points) List the points A-G that are critical points of f.

b) (2 points) List the points A-G that are critical points of f'.

c) (2 points) List the points A-G that are global maxima on the given interval $[a, b]$.

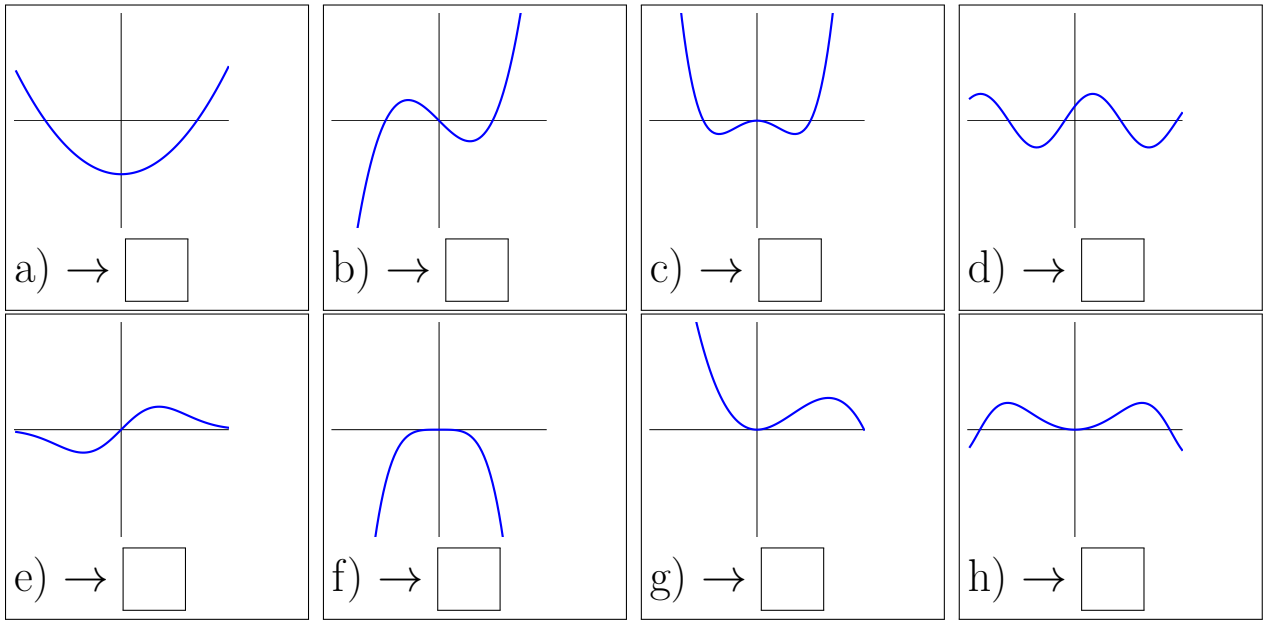
d) (2 points) List the points A-G of f that are global minima on the given interval $[a, b]$.

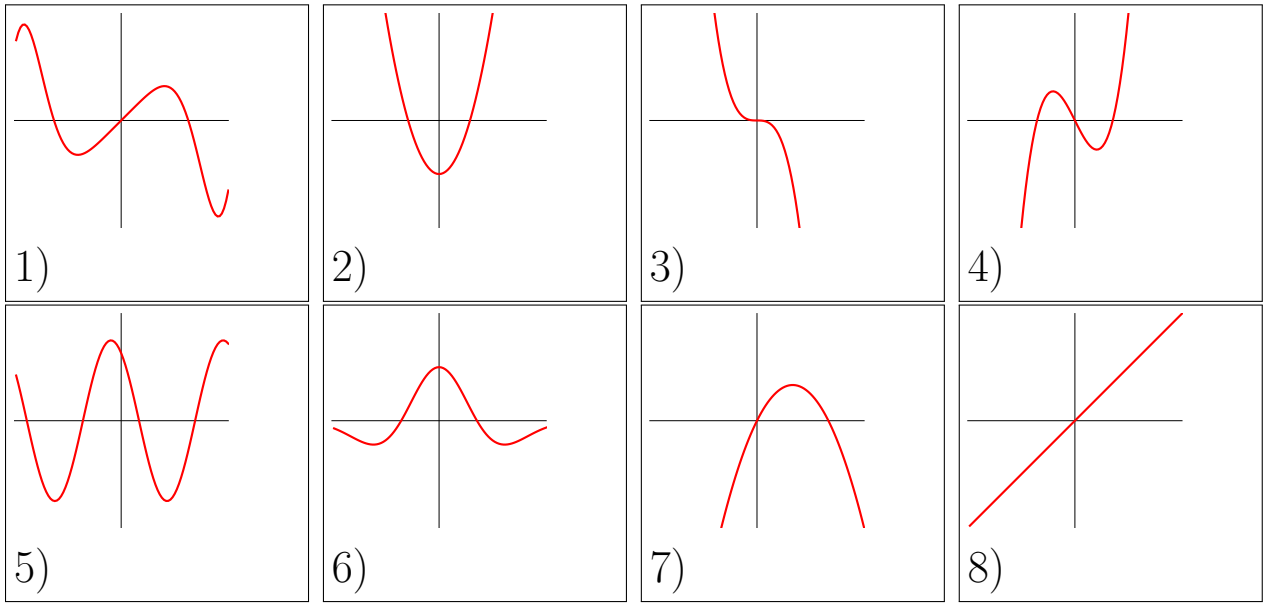
e) (2 points) Which theorem assures that there is both a global max and global min on $[a, b]$.

Solution:
a) CF, b) E, c) A, d) C , Extremal value theorem.

Problem 3) Graphing (10 points) No justifications are needed.

The functions f in a) to h) match with their derivatives 1) to 8). Find this correspondence.





Solution:

8245

6371

Problem 4) Continuity (10 points)

Make the decision "continuous on $[-5,5]$" or "not continuous on $[-5,5]$" in each of the cases a)-e) and point out any possible x values, which need special attention (*) and how you know why we have continuity there.

a) (2 points) $f(x) = \frac{x^2-9}{x-3}$

b) (2 points) $f(x) = \frac{e^{5x}-1}{e^{2x}-1}$.

c) (2 points) $f(x) = \frac{\sin^3(x)}{x^4}$

d) (2 points) $f(x) = ||3x| - 4x|$

e) (2 points) $f(x) = \sin(\sin(\frac{1}{x}))$.

Solution:

a) continuous. Problem at $x = 3$. Can simplify to $x + 3$.

b) continuous l'Hopital. Problem at $x = 0$. Use l'Hospital.

c) not continuous: the fundamental theorem of trig gives $(\frac{\sin(x)}{x})^3 \frac{1}{x}$ which is a problem at $x = 0$.

d) continuous, The function is piecewise linear

e) not continuous. There is a problem at $x = 0$. We have the devil function $\sin(\frac{1}{x})$.

(*) As usual, we extend continuity to functions for which a continuation is possible to initially not defined points, like $f(x) = (x^2 - 1)/(x - 1)$, which is considered continuous everywhere because we can fill in a function value for $x = 1$ which makes it continuous.

Problem 5) Derivatives (10 points)

Compute the derivatives. State the differentiation rules used.

a) (2 points) $f(x) = x^2 + 3x + 7$.

b) (2 points) $f(x) = \frac{1}{x} + 3e^x$

c) (2 points) $f(x) = \sin(3x) + \cos(4x)$

d) (2 points) $f(x) = \sin(x)e^x$.

e) (2 points) $f(x) = \frac{\cos(x)}{1+x}$

Solution:
a) $2x + 3$.
b) $-2/x^2 + 3e^x$
c) $3\cos(3x) - 4\sin(4x)$.
d) $\cos(x)e^x + \sin(x)e^x$. Product rule.
e) $[-\sin(x)(1+x) - \cos(x)]/(1+x)^2$.

Problem 6) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ for the following functions. As always, give details and indicate what method you are using.

a) (2 points) $f(x) = \frac{1-e^{8x}}{1-e^{4x}}$.

b) (2 points) $f(x) = \frac{1-x^2}{\cos(3x)}$.

c) (2 points) $f(x) = \frac{3x}{\ln(1+4x)}$.

d) (2 points) $f(x) = \frac{\ln(x^4)}{\ln(x^2)}$.

e) (2 points) $f(x) = x \ln|3x|$.

Solution:
a) $8/4 = 2$ Hospital
b) 1 just plug in the number $x = 0$
c) 3/4 Hospital
d) 4/2 Hospital or cancel $(4\ln(x))/(3\ln(x))$.
e) 0. This was solved by nobody. We can use Hospital after writing $\ln|3x|/(1/x)$. Which is an indefinite form ∞/∞. We can now use l'Hospital and get $(1/x)/(-1/x^2 = -x$ which has the limit $x \to 0$.

Problem 7) Functions (10 points) no explanations needed

A function is called **1-periodic** if $f(x+1) = f(x)$ for all x. A function is called **even** if $f(x) = f(-x)$ for all x. A function is called **odd** if $f(x) = -f(-x)$ for all x. A function is **invertible** from $\mathbb{R} \to \mathbb{R}$ if $f(x) = y$ has a unique solution x for every $y \in \mathbb{R}$. Invertible means that you can find an inverse function that works on the entire real line. In the following table, check each box which applies. Do not put anything in a box which does not apply.

Function	1-periodic	odd	even	invertible
$\sin(2\pi x)$				
$\cos(2\pi x)$				
$(x-1)^2$				
x^3				
e^x				

Solution:

Function	1-periodic	odd	even	invertible
$\sin(2\pi x)$	*	*		
$\cos(2\pi x)$	*		*	
$(x-1)^2$				
x^3		*		*
e^x				

Problem 8) Extrema (10 points)

a) (3 points) Find all the critical points of the function

$$f(x) = x^5 - 5x + 7 \ .$$

b) (4 points) Classify the critical points using the second derivative test.

c) (3 points) Classify the critical points using the first derivative test.

Solution:
a) We have $f'(x) = 5x^4 - 5$ which is zero at $x = 1$ or $x = -1$.
b) The second derivative is $20x^3$. So $x = 1$ is a minimum and $x = -1$ is a maximum.
c) The function f' changes from negative to positive at 1. The function f' changes from positive to negative at -1.

Problem 9) Algebra (10 points)

Simplify the following expressions. For example, $3^x 3^y$ can be written fewer letters as 3^{x+y} or $\sin^2(x)/\sin(x)$ can be simplified as $\sin(x)$. Your simplified f should be shorter than f. In the second column, fill in the derivative $f'(x)$ of the simplified expression.

	Function f	Simplified f	Derivative f'
a)	$\arccos(\cos(x^2))$		
b)	$2^{(x/\ln(2))}$		
c)	$\frac{\ln(x^4)}{\cos^2(x)+\sin^2(x)}$		
d)	$\cos(3x)\tan(3x)$		
e)	$\frac{(4x)^3}{(2x)^4}$		

Solution:
a) simplified as x^2, the derivative is $2x$.
b) simplified as e^x the derivative is e^x.
c) simplified is $4\ln|x|$, the derivative is $4/x$.
d) simplified as $\sin(3x)$ the derivative is $3\cos(3x)$.
e) simplified as $4/x$, the derivative is $-4/x^2$.

Problem 10) Linearization (10 points, 5 points each)

a) (5 points) Use linearization to estimate $996^{1/3}$.

b) (5 points) Use linearization to estimate $1004^{1/3}$.

Solution:
In both parts, take $a = 1000$ and $f(x) = x^{1/3}$ and $f'(x) = x^{-2/3}/3$. we have $f(1000) = 10$ and $f'(1000) = 1/3(10^2) = 1/300$.
a) We have the linearization $10+(1/300)(996-1000) = 10-4/300$.
b) We have the linearization $10 + (1/300)(1004 - 1000) = 10 + 4/300$.