**Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021**

# 4/9/2021: Second hourly, Practice B

**"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."**

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 10 AM on April 10th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

| | | |
|---|---|---|
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| Total: | | 100 |

Problem 1) TF questions (20 points) No justifications are needed.

1) T **F** The function $f(x) = \int x \sin(x)\, dx$ is integrated using the method of partial fractions.

**Solution:**
It is integrated using **integration by parts**!

2) **T** F For any continuous function $f$ we have $\int_0^1 3f(t)\, dt = 3\int_0^1 f(t)\, dt$.

**Solution:**
Yes this is linearity.

3) T **F** For any continuous function $\int_0^3 f(t)\, dt = 3\int_0^1 f(t)\, dt$.

**Solution:**
Looks good but is total nonsense.

4) **T** F For any continuous function $\int_0^1 1 - f(t)\, dt = 1 - (\int_0^1 f(t)\, dt)$.

**Solution:**
Because the integral over 1 can be computed directly.

5) **T** F The anti-derivative of $\tan(x)$ is $-\log(\cos(x)) + C$.

**Solution:**
Differentiate the right hand side to check.

6) **T** F The fundamental theorem of calculus implies that $\int_1^3 f'(x)\, dx = f(3) - f(1)$.

**Solution:**
Yes, this is it.

7) T F The integral $\pi \int_0^1 x^2 \, dx$ gives the volume of a cone of height 1.

**Solution:**
Yes the area of a slice is $x^2\pi$.

8) T F The anti-derivative of $1/\cos^2(x)$ is $\tan(x)$.

**Solution:**
Check.

9) T F The function $F(x) = \int_0^x \tan(t^2) \, dt$ has the derivative $\tan(x^2)$.

**Solution:**
The first derivative of $F$ is $f$.

10) T F The function $f(x) = \sin(x)/2$ on $[0, \pi]$ and $f(x) = 0$ else is a PDF

**Solution:**
We have seen this in class.

11) T F The identity $\frac{d}{dx} \int_1^2 \log(x) \, dx = \log(2) - \log(1)$ holds.

**Solution:**
We differentiate a constant.

12) T F If $f < 1$, then $\int_0^2 f(x) \, dx$ can be bigger than 1.

**Solution:**
Take $f(x) = 0.6$ for example.

13) T **F** An improper integral is an improperly defined definite indefinite integral.

**Solution:**
If you marked this true, you must have been properly drunk or behaved improperly.

14) **T** F The anti derivative $F(x)$ of $f(x)$ satisfies $F'(x) = f(x)$.

**Solution:**
This is the fundamental theorem of calculus

15) **T** F A parameter value $c$ for which the number of minima are different for parameters smaller or larger than $c$ is called a catastrophe.

**Solution:**
This is a definition.

16) T **F** If $f$ is unbounded at 0, then $\int_0^1 f(x)\,dx$ is infinite.

**Solution:**
The function $\sqrt{x}$ was a counter example.

17) T **F** If $f(-1) = 0$ and $f(1) = 1$ then $f' = 2$ somewhere on $(-1, 1)$.

**Solution:**
This is close to the intermediate value theorem.

18) **T** F The anti-derivative of $\log(x)$ is $x\log(x) - x + C$, where log is the natural log.

**Solution:**
You might not have known this by heart, but you can check it!

19) T F The sum $\frac{1}{n}[(\frac{0}{n})^2 + (\frac{1}{n})^2 + ... + (\frac{n-1}{n})^2]$ converges to 1/3 in the limit $n \to \infty$.

**Solution:**
It is a Riemann sum.

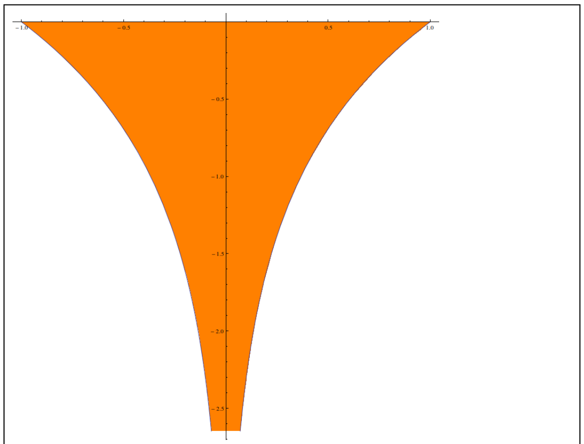
20) T F The **improper integral** $\int_1^\infty \frac{1}{x^2}\, dx$ represents a finite area.

**Solution:**
We have no problem at infinity.

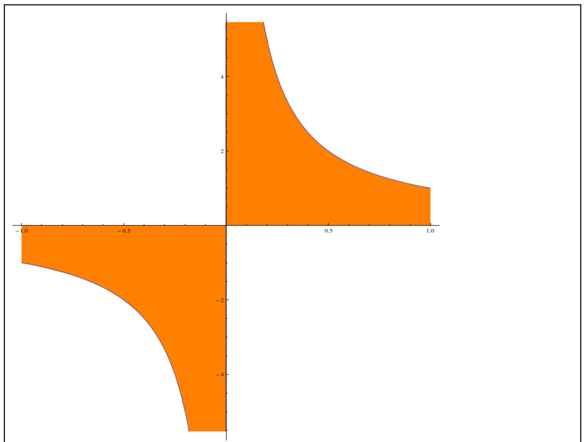
| Problem 2) Matching problem (10 points) No justifications are needed. |
|---|

a) (4 points) Match the following integrals with the regions and indicate whether the integral represents a finite area.

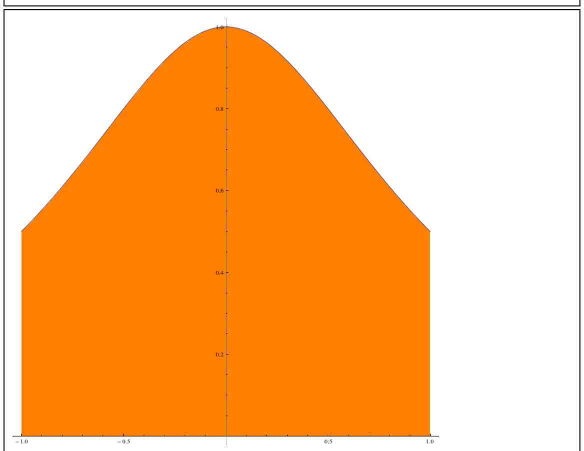
| Integral | Fill in 1-4 | Finite? |
|---|---|---|
| $\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx$ | | |
| $\int_{-1}^{1} \frac{1}{x}\, dx$ | | |
| $\int_{-1}^{1} \frac{1}{1+x^2}\, dx$ | | |
| $\int_{-1}^{1} \log\|x\|\, dx$ | | |



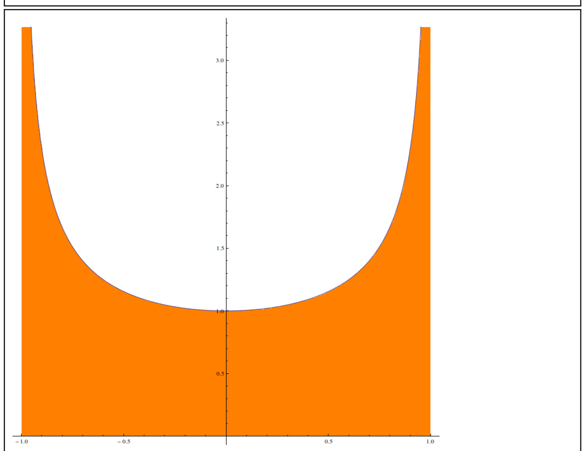
1)



2)



3)



4)

b) (6 points) Which of the following properties are always true. This means which are true for all choices of continuous functions and all choices of $a, b, c$.

| Identity | Check if true |
|---|---|
| $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ | |
| $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b f(x) + g(x)\,dx$ | |
| $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ | |
| $\int_a^b f(x)^2\,dx = (\int_a^b f(x)\,dx)^2$ | |
| $\int_a^a f(x)\,dx = 0$ | |
| $\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$ | |

**Solution:**

| Integral | Fill in 1-4 | Finite? |
|---|---|---|
| $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx$ | 4 | * |
| $\int_{-1}^1 \frac{1}{x}\,dx$ | 2 | |
| $\int_{-1}^1 \frac{1}{1+x^2}\,dx$ | 3 | * |
| $\int_{-1}^1 \log\lvert x\rvert\,dx$ | 1 | * |

| Identity | Check if true |
|---|---|
| $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ | * |
| $\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = \int_a^b f(x) + g(x)\,dx$ | * |
| $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ | * |
| $\int_a^b f(x)^2\,dx = (\int_a^b f(x)\,dx)^2$ | |
| $\int_a^a f(x)\,dx = 0$ | * |
| $\int_a^b f(x)\,dx = \int_b^a f(x)\,dx$ | |

Problem 3) (10 points)

Fill in the missing part into the empty box to make a true statement:

a) (2 points)

$\frac{d}{dx}\int_0^x f(t)\,dt =$ [ ] by the **fundamental theorem of calculus**.

b) (2 points)

$\int_0^x f(t)\ dt =$ [ ] by the **fundamental theorem of calculus**.

c) (2 points)

The **method of** [ ] writes $f(x) = 1/((x-6)(x+9))$ as $A/(x-6) + B/(x+9)$ and fixes the constants $A, B$.

d) (2 points)

A **probability distribution** satisfies $\int_{-\infty}^{\infty} f(x)\ dx = 1$ and [ ] for all $x$.

e) (2 points)

For an improper integral $\int_a^b f(x)\ dx$, either $a = \infty$ or $b = \infty$ or $f$ is [ ] on $[a, b]$.

**Solution:**
a) $f(x)$
b) $F(x) - F(0)$ if $F$ is the anti-derivative.
c) Method of partial fractions.
d) $f \geq 0$
e) unbounded or discontinuous

Problem 4) Area computation (10 points)

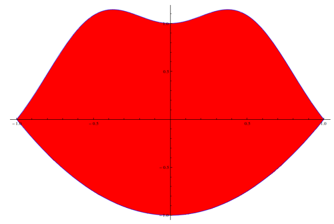
The region enclosed by the graphs of

$$f(x) = x^2 - 1$$

and

$$g(x) = 1 - x^2 + (1 - \cos(2\pi x))/6$$

models of the lips of **Rihanna**. Find the area.

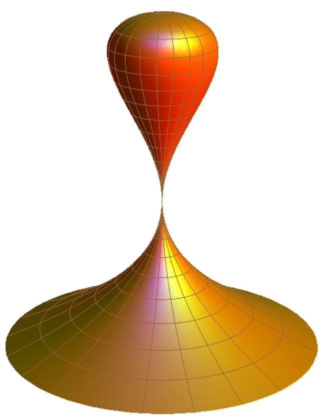



**Solution:**
The function $g$ is above. The two graphs intersect at $-1$ and 1. We have

$$\int_{-1}^{1} 1 - x^2 + (1 - \cos(2\pi x))/6 - (x^2 - 1)\, dx = 2x - 2x^3/3 + 2x/6 - \sin(2\pi x)/(12\pi)|_{-1}^{1} = 3 .$$

The answer is $\boxed{3}$.

Problem 5) Volume computation (10 points)



The **kiss** is a solid of revolution for which the radius at height $z$ is

$$z^2\sqrt{1-z}$$

and where $-1 \leq z \leq 1$. What is the volume of this solid? The name "kiss" is the official name for this quartic surface. Indeed, the top part has the shape of a **Hershey Kiss**. P.S. Creative **"exam product placement"** like this has been invented and patented by Oliver himself ...

**Solution:**
The area is $z^4(1-z)\pi$. We integrate this from $-1$ to 1 we first expand and write $\pi \int_{-1}^{1} z^4 - z^5\, dz = 2\pi/5$. P.S. Oliver messed with you in the PS part. But Oliver (who writes this), always lies.

Problem 6) Definite integrals (10 points)

Evaluate the following definite integrals. Each of the problems produces a numerical answer.

a) (2 points) $\int_0^1 \sqrt{1+x}\, dx$.

b) (2 points) $\int_1^2 \frac{1}{9-x^2}\, dx$

c) (2 points) $\int_2^e x \log(x)\, dx$

d) (2 points) $\int_1^2 \frac{x^2}{\sqrt{9-x^3}}\, dx$.

e) (2 points) $\int_0^\pi x^3 \sin(x)\, dx$

**Solution:**
a) Inegrate directly $(2/3)(\sqrt{8}-1)$
b) Use the method of partial fractions $\log(5)/6 - \log(2)/6$
c) Use integration by parts $\log\left(\frac{81\sqrt{3}}{4}\right) - \frac{5}{4}$.
d) Use substitution. The anti derivative is $-\log(9-x^3)/3$. Plugging in gives $\log(2)$. e) Use integration by parts three times (Tic Tac Toe). The answer is $\pi(\pi^2-6)$.

Problem 7) Anti derivatives (10 points)

Find the following anti-derivatives:

a) (2 points) $\int 23e^{23x} - x^{23}\, dx$

b) (2 points) $\int \frac{2}{x+3} + x^{1/23}\, dx$

c) (2 points) $\int \frac{23x}{1+x^2} + 23\tan(x)\, dx$

d) (2 points) $\int \log(x)^2 dx$

e) (2 points) $\int \cos^2(3x)\, dx$

Jim Carrey in the movie "The number 23"

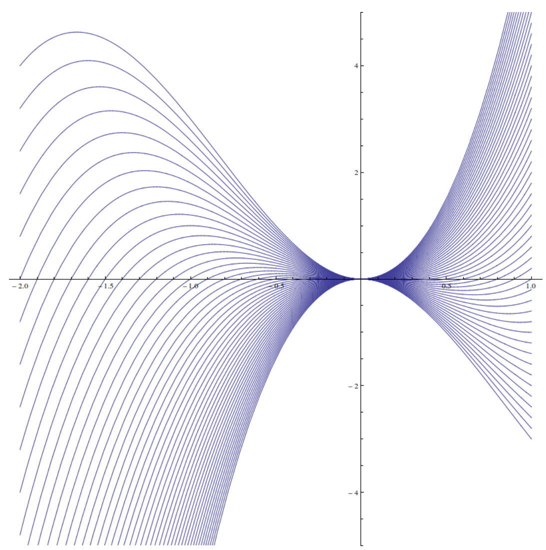
**Solution:**
a) Direct $e^{23x} - x^{24}/24 + c$.
b) Also direct $2\log(x+3) + (23/24)x^{24/23} + c$.
c) Substitution $u = 1 + x^2$ works for the first and $u = \cos(x)$ for the second. We have $23\log(1+x^2)/2 - 23\log(\cos(x)) + c$.
d) Integration by parts twice $2x - 2x\log(x) + x\log(x)^2 + c$.
e) Double angle formula $\sin(6x)/12 + x/2 + c$.

Problem 8) PDF's and CDF's (10 points)

Verify that the function $f(x) = \exp(-|x|)/2$ is a PDF.

**Solution:**
Yes, it is positive everywhere and $\int_0^\infty e^{-|x|}\ dx = 1$ and $\int_{-\infty}^0 e^{-|x|}\ dx = 1$ so that $\int_{-\infty}^\infty f(x)\ dx = 1$.

Problem 9) Catastrophes (10 points)



We look at the one-parameter family of functions $f_c(x) = 2x^3 + cx^2$, where $c$ is a parameter.

a) (2 points) Find the critical points of $f_3(x)$.

b) (2 points) Find the critical points of $f_{-3}(x)$.

c) (2 points) Check that 0 is always a critical point.

d) (2 points) For which $c$ is 0 a minimum?

e) (2 points) For which $c$ does the catastrophe occur?

**Solution:**
a) $f'(x) = 6x^2 + 2cx = 2x(3x + c) = 0$ means either $x = 0$ or $x = -1$.
b) $f'(x) = 6x^2 - 6x = 0$ means either $x = 0$ or $x = 1$.
c) $f'(x) = 5x^2 - 2cx$ is always 0 independent of $c$.
d) $f''(0) = 2c = 2c$. We see that for $c > 0$ we have a minimum and for $c < 0$ a maximum.
e) At $c = 0$ the nature of the critical point changes. The parameter $c = 0$ is the catastrophe.