Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

4/9/2021: Second hourly

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as a PDF, handwritten in one file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids or online tools or external information are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 10 AM on April 10th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) **T** F

The function $f(x) = \frac{e^{-(x^2/2)}}{\sqrt{2\pi}}$ is a PDF called the normal distribution.

Solution:
Yes, this is an important function.

2) T **F**

The method of partial fractions is based on the chain rule.

Solution:
No, it is algebra.

3) T **F**

$\int \log(x)\, dx = \frac{1}{x} + C$.

Solution:
Wrong way

4) **T** F

If $0 \leq f(x) \leq 1$, then $0 \leq \int_0^1 f(x)\, dx \leq 1$

Solution:
Think about the area. It is in a square of area 1.

5) T **F**

A PDF is the anti-derivative of the CDF.

Solution:
Wrong way.

6) **T** F

If $f(x) = 1$ everywhere and $\int_a^b f(x)\, dx = 0$ then $a = b$.

Solution:
Yes, there is no constant.

7) T **F** If f is continuous, then $\int_a^b f(-x)\, dx = -\int_a^b f(x)\, dx$.

Solution:
This is not true in general. It would be true for odd functions.

8) **T** F If F is a cumulative distribution function and $a < b$, then $F(b) - F(a)$ is the probability that the data are in the interval $[a, b]$.

9) T **F** The function $e^x \cos(x)$ needs to be integrated by partial fractions.

Solution:
No, it is integration by parts

10) **T** F The fundamental theorem of calculus implies $\int_a^b f''(x)\ dx = f'(b) - f'(a)$ if f'' is a continuous function.

Solution:
Yes, this usually just appears for $f(x)$ and not $f'(x)$.

11) T **F** The improper integral $\int_0^1 \frac{1}{x^{3/2}}\, dx$ is finite.

Solution:
It is an improper integral for which the area does not exist.

12) **T** F Much of catastrophe theory has been developed and propagated by René Thom.

Solution:
Yes that is a name to remember.

13) T **F** $\int_1^2 \log(5x)dx$ is one fifth of $\int_1^2 \log(x)\, dx$.

Solution:
Make a substitution

14) **T** F A CDF is monotonically increasing in the sense $F(x) \leq F(y)$ if $x \leq y$.

Solution:
Its derivative is non-negative.

15) T **F** If $\lim_{x\to\infty} f(x)^2 = 0$, then $\int_1^\infty f(x)\, dx$ is finite.

Solution:
A counter example is $f(x) = 1/\sqrt{x}$.

16) T **F** An integral is called improper if it does not have a finite value.

Solution:
Improper integrals can be finite.

17) T **F** The anti-derivative of $\arctan(x)$ is $1/(1+x^2)+C$.

Solution:
The derivative of $\arctan(x)$ is $1/(1+x^2)$, not tan itself.

18) **T** F Tic-Tac-Toe integration method allows to integrate functions like $x^{10}e^x$.

Solution:
Yes. It is a multiple integration method.

19) T F — Gabriel's trumpet has infinite volume and finite surface area.

Solution:
It has finite volume π.

20) T F — The function $f(x) = e^{-x}$ for positive x and $f(x) = 0$ else is the exponential distribution.

Solution:
It is e^{-x} not e^x.

Problem 2) Theorems (10 points) No justifications needed.

Fill in the missing part into the empty box to make a true statement. Read the statements carefully.

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a) $\frac{d}{dx} \int_x^1 f(t)\, dt = f(1) -$ [ ] by the **fundamental theorem of calculus**.

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b) Applying the iteration $T(x) = x - f(x)/f'(x)$ is called the [ ].

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c) A **probability distribution** is a piece-wise continuous function which satisfies $\int_{-\infty}^{\infty} f(x)\, dx = 1$ and [ ].

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d) The **improper integral** $\int_0^1 \frac{1}{x^p}\, dx$ converges, if p is contained in the open interval [ ].

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e) Assume $f_c(x)$ is a **family of functions** such that for $c < 0$, there are exactly 3 minima and for $c > 0$ there are exactly 2 minima, then c is called a [ ].

Solution:

a) $f(x) + f(1)$.

b) Newton step.

c) $f \geq 0$.

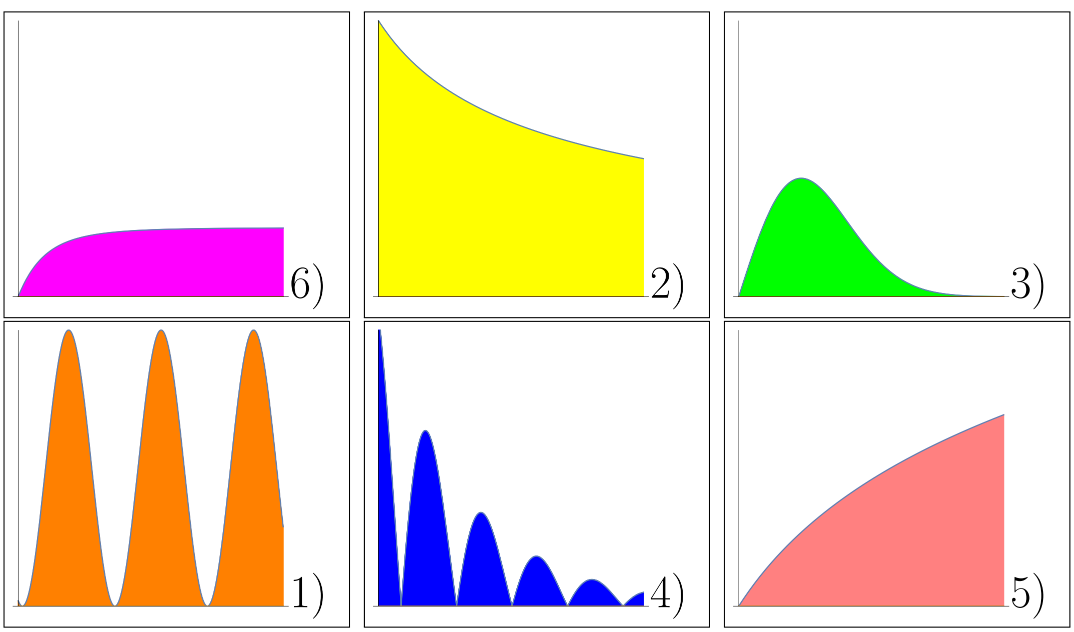
d) $p \in (-\infty, 1)$.

e) catastrophe.

Problem 3) Matching problem (10 points)

Match the following integrals with parts of the regions and indicate whether the integral is convergent or divergent.

Integral	1-6	Convergent	Divergent
$\int_1^\infty \sin^2(3x)\ dx$			
$\int_1^\infty \frac{1}{x^{1/2}}\ dx$			
$\int_1^\infty (x-1)e^{-(x-1)^2}\ dx$			
$\int_1^\infty \log(x)\ dx$			
$\int_1^\infty \frac{x^4-1}{x^4+1}\ dx$			
$\int_1^\infty 3\|\sin(5x)\|e^{-x}\ dx$			



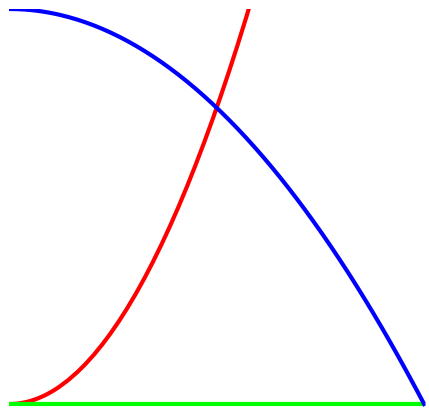
Solution:

1,2,3,5,6,4.

Divergent, Divergent, Convergent, Divergent, Divergent, Convergent

Problem 4) Area computation (10 points)

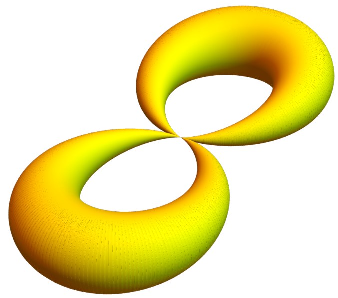
Find the **area of the triangular region** sandwiched between the graphs of $f(x) = 3x^2$ and $g(x) = 0$ and $h(x) = 4 - x^2$. Document your work.



Solution:
Split the integral up $\int_0^1 3x^2 \; dx + \int_1^2 4 - x^2 \; dx$. The answer is $\boxed{8/3}$.

Problem 5) Volume computation (10 points)

A solid has a cross section area of $A(x) = \pi \sin^2(x)$, where x goes from 0 to 2π. What is the **volume** V of that solid? As mentioned in class, the **Archimedes formula** still works also for the volume of the solid if it is bent around as long as the bending does not change the area of the cross section. The picture shows such a solid.



Solution:
The volume is

$$\int_0^{2\pi} A(x)\, dx = \pi \int_0^{2\pi} \sin^2(x)\, dx = \pi^2 \; .$$

Problem 6) Definite integrals (10 points, 2 points each)

Compute the following definite integrals. State the integration steps and names.

a) $\int_0^1 2xe^{x^2+3}\, dx$
b) $\int_0^1 4x^3/(1+x^4)\, dx$
c) $\int_0^1 \frac{1}{(x-5)(x-3)}\, dx$
d) $\int_0^1 \log^2(1+x)\, dx$
e) $\int_0^1 xe^{1+x}\, dx$

Solution:

a) Use substitution $u = x^2 + 3, du = 2dx$. The answer is $\boxed{e^4 - e^3}$.

b) Use substitution $u = 1 + x^2, du = 2xdx$ again. The answer is $\boxed{\log(2)}$.

c) Partial fractions $\log(6)/2 - \log(5)/2$.

d) Integrate by parts $\log(4) - 2)$.

e) Use integration by parts to get $F(x)_0^1 = e^{1+x}(x-1)|_0^1$. Evaluation gives $\boxed{E}$.

Problem 7) Anti derivatives (10 points, 2 points each)

Solve the indefinite integrals. State the integration method in each case.

a) $\int \frac{\log(x)}{x^2}\, dx$
b) $\int \cos^2(x) - 3\sin^2(x)\, dx$
c) $\int \frac{1}{x^2-8x+12}\, dx$
d) $\int (x+1)^3 e^x\, dx$
e) $\int \frac{3x^2}{(1+x^6)}\, dx$

Solution:

a) Parts, $(-\log(x) - 1)/x$.

b) Double angle, $-x + \sin(2x)$.

c) partial fraction, $\log(6-x)/4 - \log(2-x)/4$

d) Use Tic-Tac-Toe: parts $e^x(x^3 + 3x - 2)$.

e) Substitution $\arctan(x^3) + C$

Problem 8) PDF's and CDF's (10 points)

We look at the function $f(x)$ which is $x^7/32$ for $0 \leq x \leq 2$ and $f(x) = 0$ else.

a) (4 points) Verify that $f(x)$ is a PDF.

b) (3 points) If data are distributed according to $f(x)$, what is the probability of the data to be in the interval $[a, b] = [0, 1]$?

c) (3 points) What is the mean of the distribution f?

Solution:
a) Because the function is zero outside the interval $[0, 2]$ we can integrate $\int_0^2 f(x)\, dx$. The result is 1. b) It is $\int_0^1 f(x)\, dx = 1/256$. c) It is $\int_0^2 x f(x)\, dx = \int_0^2 x^8/32\, dx = 16/9$.

Problem 9) Catastrophes (10 points)

Consider the family of functions $f(x) = x^3/3 + cx$ on the real line.

a) (4 points) Find all critical points of f for $c < 0$ and determine the stable ones or indicate there are none.

b) (4 points) Find all critical points of f for $c > 0$ and determine the stable ones or indicate there are none.

c) (2 points) For which value of c does a catastrophe occur?

Solution:
a) The critical points are $x^2+c=0$ which means $x=\sqrt{-c}$. There are no critical points for $c>0$ and two different critical points for $c<0$.
b) The second derivative is $2x$ which is negative for $x<0$ and positive for $x>0$. The point $\sqrt{-c}$ is a local minimum for $c<0$. There is exactly one critical point which is a minimum for $c<0$.
c) The catastrophe appears at the parameter $c=0$ because a local minimum, present for $c<0$ disappears for $c\geq 0$.