Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

4/9/2021: Second hourly, Practice E

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 10 AM on April 10th. Do not communicate with anybody related to the class during the exam period and with nobody at all about the exam.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1)	T	F	The formula $\int_0^x f''(x)\, dx = f'(x) - f'(0)$ holds.
2)	T	F	The area of the upper half disc is the integral $\int_{-1}^1 \sqrt{1-x^2}\, dx$
3)	T	F	If the graph of the function $f(x) = x^2$ is rotated around the interval $[0,1]$ in the x axes we obtain a solid with volume $\int_0^1 \pi x^4\, dx$.
4)	T	F	The function $f(x) = e^x$ is the only anti derivative of e^x.
5)	T	F	If f has a critical point at 1, then $F(x) = \int_0^x f(t)\, dt$ has an inflection point at 1.
6)	T	F	Catastrophes are parameter values c for a family of functions $f_c(x)$, for which a local minimum of f_c disappears.
7)	T	F	The volume of a cylinder of height and radius 1 minus the volume of a cone of height and radius 1 is half the volume of a sphere of radius 1.
8)	T	F	The function $f(x) = 1/(\pi\sqrt{1-x^2})$ for $-1 < x < 1$ and $f(x) = 0$ else is a PDF
9)	T	F	The improper integral $\int_0^1 x^{1/3}\, dx$ is finite.
10)	T	F	Integrals are linear: $\int_0^x f(t) + g(t)\, dt = \int_0^x f(t)\, dt + \int_0^x g(t)\, dt$.
11)	T	F	The function $\text{Li}(x) = \int_2^x dt/\log(t)$ has an anti-derivative which is a finite construct of trig functions.
12)	T	F	There is a region enclosed by the graphs of x^5 and x^6 which is finite and positive.
13)	T	F	The integral $\int_{-1}^1 1/x^4\, dx = -1/(5x^5)\|_{-1}^1 = -1/5 - 1/5 = -2/5$ is defined and negative.
14)	T	F	Gabriel's trumpet has finite volume but infinite surface area.
15)	T	F	A function $f(x)$ is a probability density function, if $f(x) \geq 0$ and $\int_{-\infty}^{\infty} f(x)\, dx = 1$.
16)	T	F	The mean of a function on an interval $[a,b]$ is $\int_a^b f(x)\, dx$.
17)	T	F	The cumulative probability density function is an anti-derivative of the probability density function.
18)	T	F	The integral $\int_{-\infty}^{\infty} (x^2 - 1)\, dx$ is finite.
19)	T	F	The total prize is the derivative of the marginal prize.
20)	T	F	The acceleration is the anti-derivative of the velocity.

Problem 2) Matching problem (10 points) No justifications are needed.

Match the following functions with their anti derivatives. Of course only 6 of the 30 functions will appear.

Function	Anti-derivative Enter 1-30
$\cos(3x)$	
$\sin(3x)$	
$3x$	

Function	Anti-derivative Enter 1-30
$1/(3x)$	
$\tan(3x)$	
$1/(1+9x^2)$	

1) $\sin(3x)$
2) $-\sin(3x)/3$
3) $\sin(3x)/3$
4) $-3\sin(3x)$
5) $3\sin(3x)$

6) $\cos(3x)$
7) $-\cos(3x)/3$
8) $\cos(3x)/3$
9) $-3\cos(3x)$
10) $3\cos(3x)$

11) $\log(x)/3$
12) $1/(3-x)$
13) $1/(3x)$
14) $\log(x/3)$
15) $-1/(3x^2)$

16) $3x^2$
17) $x^2/2$
18) $3x^2/2$
19) 3
20) x^2

21) $\arctan(3x)/3$
22) $3\arctan(3x)$
23) $1/(1+9x^2)$
24) $3/(1+9x^2)$
25) $-3/(1+x^2)$

26) $1/\cos^2(3x)$
27) $\log(\cos(3x))$
28) $-\log(\cos(3x))/3$
29) $\log(\cos(3x))/3$
30) $3/\cos^3(3x)$

Problem 3) Matching problem (10 points) No justifications are needed.

Which of the following formulations is a Riemann sum approximating the integral $\int_0^3 f(x)\,dx$ of $f(x) = x^2$ over the interval $0, 3$.

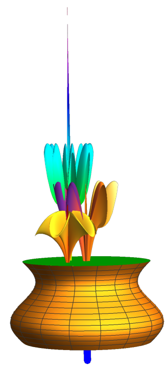
Sum	Check if this is the Riemann sum.
$n\sum_{k=0}^{n-1}(3k/n)^2$	
$\frac{1}{n}\sum_{k=0}^{n-1}(3k/n)^2$	
$n\sum_{k=0}^{3n-1}(k/n)^2$	
$\frac{1}{n}\sum_{k=0}^{3n-1}(k/n)^2$	

Problem 4) Area computation (10 points)

Find the area of the region enclosed by the three curves $y = 6 - x^2, y = -x$ and $y = x$ which is above the x axes.

Problem 5) Volume computation (10 points)

Emma Woodhouse grows plants in a pot which is a rotationally symmetric solid for which the radius at position x is $5 + \sin(x)$ and $0 \leq x \leq 2\pi$. Find the volume of the pot.



Problem 6) Definite integrals (10 points)

Find the following definite integrals

a) (5 points) $\int_1^2 x^2 \sin(\pi x)\, dx$.

b) (5 points) $\int_1^3 x^2 \cos(x^3 + 2)\, dx$

Problem 7) Anti-derivatives (10 points)

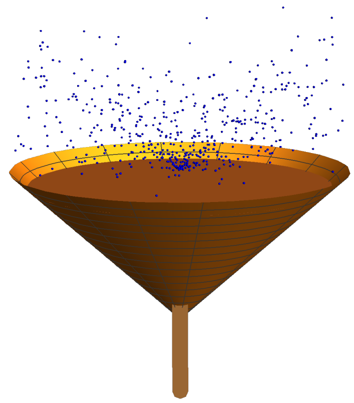
Find the following anti-derivatives

a) (5 points) $\int \frac{3}{\sqrt{1-x^2}} + x^4 + \frac{1}{1+x^2}\, dx$

b) (5 points) $\int \frac{1}{(x-2)(x+4)} + \frac{2}{x-1}\, dx$

Problem 8) Chain rule and volume (10 points)

A coffee machine has a filter which is a cone of radius z at height z. If the coffee covers everything until height t, let $V(t)$ denote the volume and $z(t)$ the height at time t. Coffee spills out at a rate of $V'(t) = 1$ cubic centimeter per second. How fast does the water level sink at height $z = 10$? To solve this, note that the chain rule equates $d/dtV(t)$ with $\pi z^2 z'(t)$.



Problem 9) PDF's and CDF's (10 points)

Assume we know that the CDF is given by $f(x) = 1/2 + \arctan(x)/\pi$.
Determine the corresponding PDF.

Problem 10) Improper integrals (10 points)

Evaluate the following improper integrals or state that they do not exist

a) (3 points) $\int_1^\infty 1/\sqrt{x}\, dx$.

b) (2 points) $\int_0^1 \sqrt{x}\, dx$.

c) (3 points) $\int_0^\infty 2xe^{-x^2}\, dx$.

d) (2 points) $\int_0^\infty \frac{1}{x}\, dx$