Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

3/5/2021: First hourly Practice E

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 6 AM on March 6th.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T **F** The function arcsin(x) is defined as $1/\sin(x)$.

Solution:
The arcsin function is the inverse not the reciprocal of $\sin(x)$.

2) T **F** The function $f(x) = \sin(1/x^2)$ can be defined at 0 so that it becomes a continuous everywhere on the real line.

Solution:
This is the prototype oscillatory singularity.

3) T **F** The function $x/\sin(x)$ can be defined at $x = 0$ so that it becomes a continuous function on the real line.

Solution:
The value is 1 at 0 by the fundamental theorem of trigonometry. It can not be made continuous on the entire real line because at $\pi, 2\pi$ etc the function can not be saved.

4) **T** F The function $f(x) = \sin^2(x)/x^2$ has the limit 1 at $x = 0$.

Solution:
Yes, it is the square or the *sinc* function.

5) T **F** The function $f(x) = 1/\log|x|$ has the limit 1 at $x = 0$.

Solution:
l'Hospital gives 0, not 1.

6) **T** F The function $f(x) = (1+h)^{x/h}$ has the property that $Df(x) = [f(x+h) - f(x)]/h = f(x)$.

Solution:
We have seen that several time in this course and done in the homework.

7) | T | **F** | $\cos(3\pi/2) = 1$.

Solution:
Draw the circle. The angle $3\pi/2$ corresponds to 270 degrees. The cosine is the x value and so zero.

8) | **T** | F | If a function f is continuous on the interval $[3, 10]$, then it has a global maximum on this interval.

Solution:
This is a consequence of the extreme value theorem.

9) | T | **F** | The reciprocal rule assures that $d/dx(1/g(x)) = 1/g(x)^2$.

Solution:
The minus sign is missing as well as the factor $g'(x)$.

10) | **T** | F | If $f(0) = g(0) = f'(0) = g'(0) = 0$ and $g''(0) = f''(0) = 1$, then $\lim_{x\to 0}(f(x)/g(x)) = 1$

Solution:
This is a consequence of l'Hospital's rule when applied twice.

11) | **T** | F | An inflection point is a point where the function $f''(x)$ changes sign.

Solution:
This is a definition.

12) T F

If $f''(x) > 0$ then f is concave up at x.

Solution:
The slope of the tangent increases which produces a concave up graph. One can define concave up with the property $f''(x) > 0$

13) T F

The chain rule assures that $d/dx f(g(x)) = f'(x)g'(x)$.

Solution:
This is not true. We have $f'(g(x))$ in the first factor.

14) T F

The function $f(x) = 1/x + \log(x)$ is continuous on the interval $[1, 2]$.

Solution:
While there is a problem at 0, everything is nice and dandy at $[1, 2]$.

15) T F

The function $(e^x - 1)/\cos(x)$ defines an indefinite form at ∞.

Solution:
No, $\cos(x)$ does not go to 0.

16) T F

The graph of the function $f(x) = x/(1 + x^2)$ has slope 1 at 0.

Solution:
$f'(x) = 1/(1 + x^2) - 2x^2/(1 + x^2)^2$. This is 1 for $x = 0$.

17) T F

There is a differentiable function for which $f'(0) = 0$ but for which 0 is not a local extremum.

Solution:
Take $f(x) = x^3$.

18) T **F** The second derivative test assures that $x = p$ is a local minimum if $f'(p) = 0$ and $f''(p) < 0$.

Solution:
It is $f''(x) > 0$.

19) **T** F The identity $(x^7 - 1)/(x - 1) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ holds for all $x \neq 1$.

Solution:
Multiply out to see it.

20) T **F** The slope of the tangent at a point $(x, f(x))$ of the graph of a differentiable function f is equal to $1/f'(x)$.

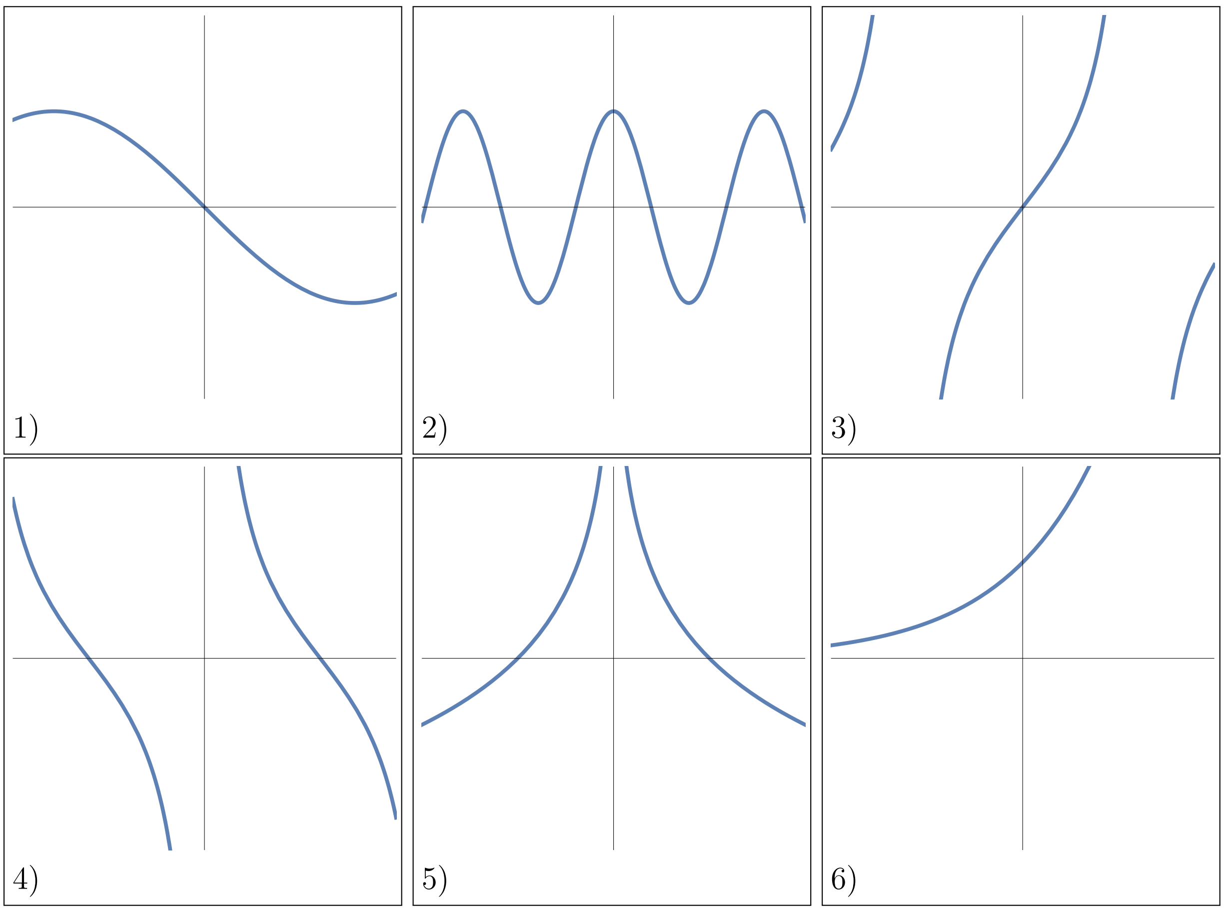
Solution:
The slope is $f'(x)$ not $1/f'(x)$.

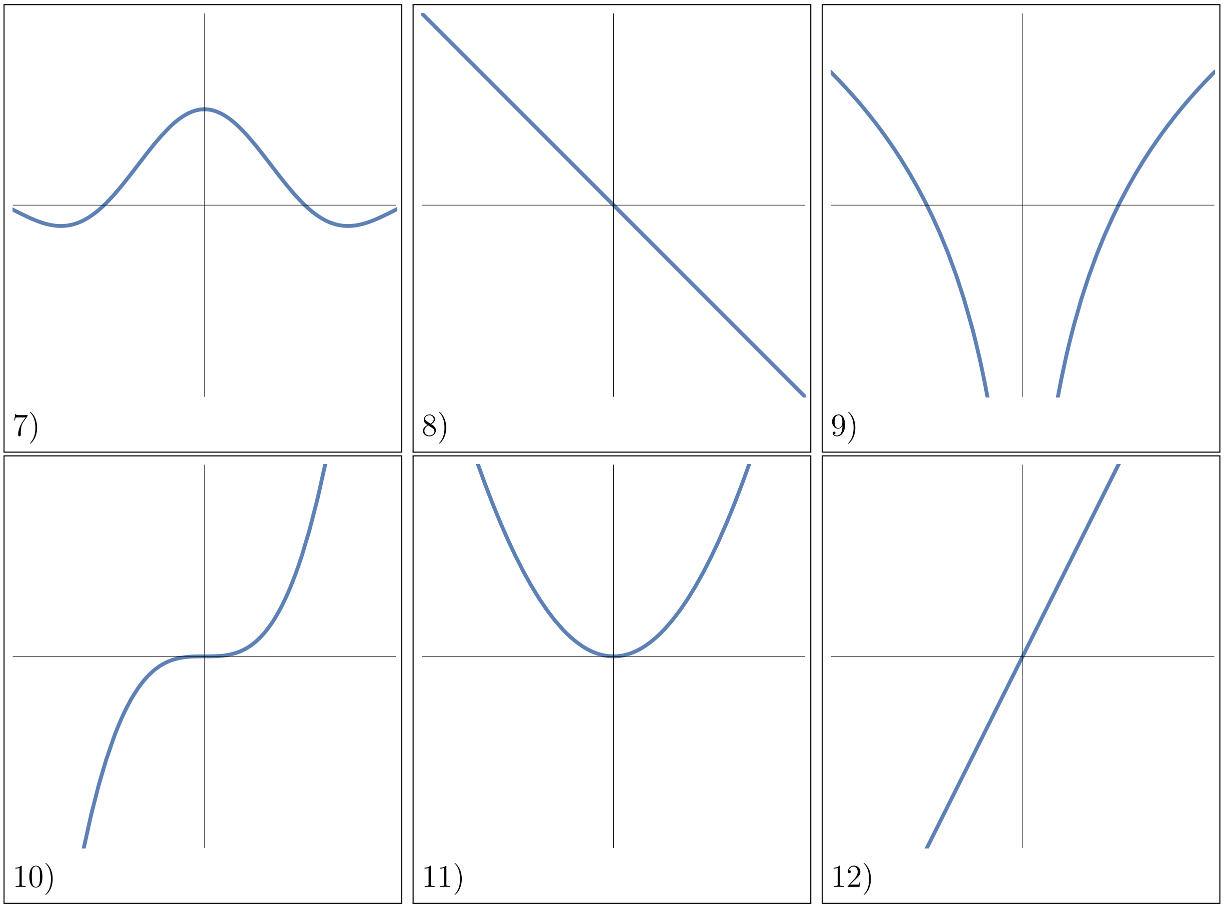
Problem 2) Matching problem (10 points) No justifications are needed.

Match the functions with the graphs. Naturally, only 10 of the 12 graphs will appear.

Function	Enter 1-12
$\cot(x)$	
$\cos(2x)$	
$2x$	
$\tan(x)$	
$\log(1/\lvert x\rvert)$	

Function	Enter 1-12
x^2	
$\exp(x)$	
$-\sin(x)$	
x^3	
$\mathrm{sinc}(x)$	





Solution:

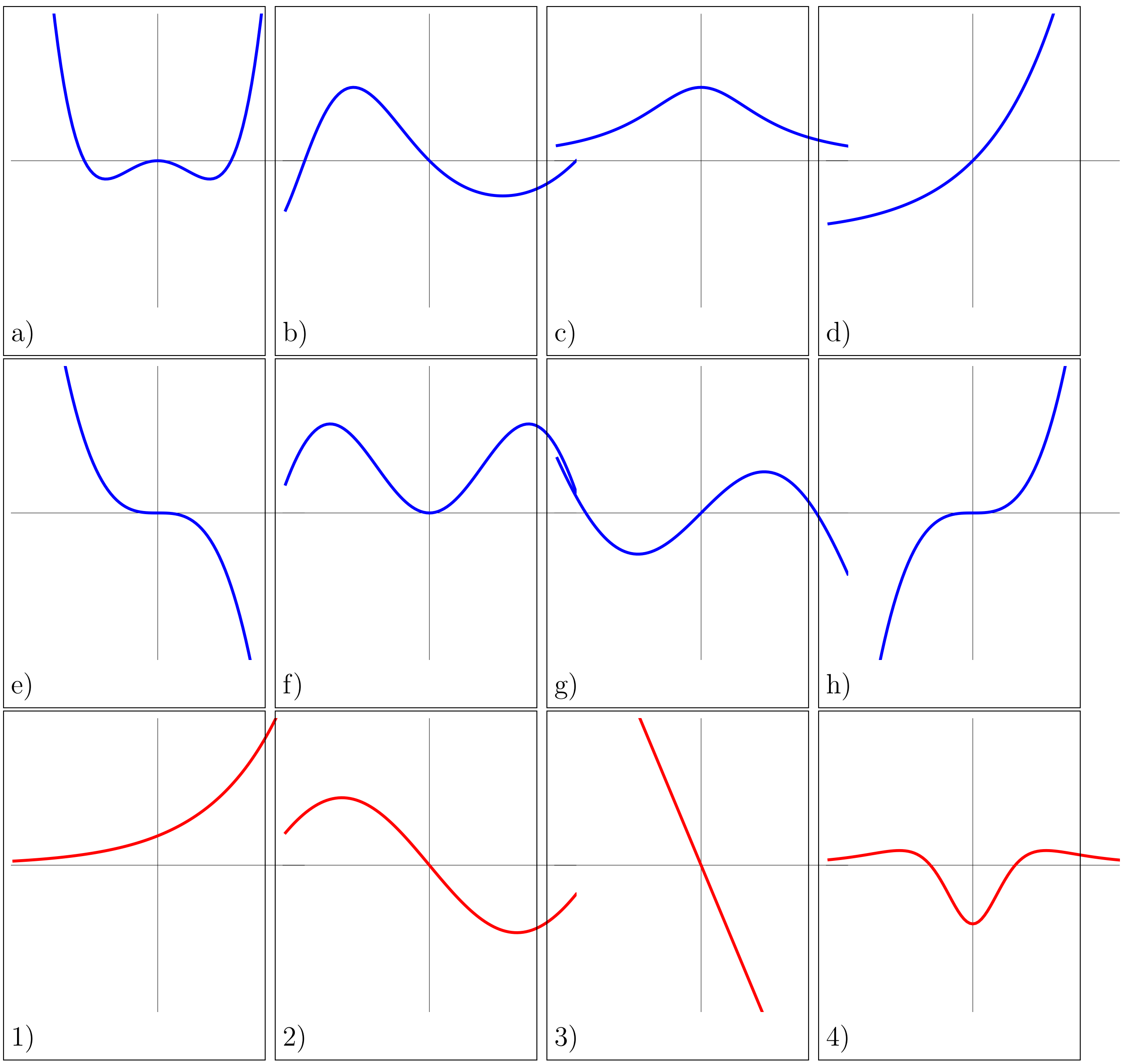
Function	Enter 1-12
$\cot(x)$	4
$\cos(2x)$	2
$2x$	12
$\tan(x)$	3
$\log(1/\|x\|)$	5

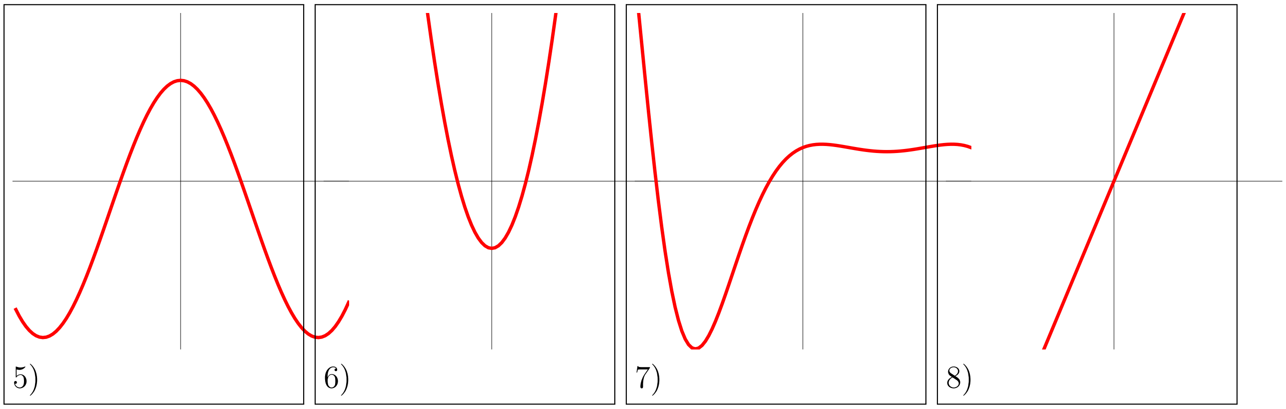
Function	Enter 1-12
x^2	11
$\exp(x)$	6
$-\sin(x)$	1
x^3	10
$\text{sinc}(x)$	7

Problem 3) Matching problem (10 points) No justifications are needed.

Match the graph of the functions f in $a) - h)$ with the second derivatives f'' in 1)-8).

Function	Second derivative (Enter 1- 8 here)
a)	
b)	
c)	
d)	
e)	
f)	
g)	
h)	





Solution:

Function	Solution
a)	6 or 4
b)	7
c)	4 or 6
d)	1
e)	3
f)	5
g)	2
h)	8

Both 6,7,4 and 4,7,6 are possible.

Problem 4) Continuity (10 points)

Some of the following functions might a priori not be defined yet at the point a. In each case, decide whether f can be made a continuous function by assigning a value $f(a)$ at the point a. If no such value exist, state that the function is not continuous.

a) (2 points) $f(x) = \frac{(x^3-1)}{(x-1)}$, at $x = 1$

b) (2 points) $f(x) = \sin(\frac{1}{x}) + \cos(x)$, at $x = 0$

c) (2 points) $f(x) = \sin(\frac{1}{\log(|x|)})$, at $x = 0$

d) (2 points) $f(x) = \log(|\sin(x)|)$, at $x = 0$

e) (2 points) $f(x) = \frac{(x-1)}{x}$, at $x = 0$

Solution:
a) Heal the function by dividing out $(x-1)$. For $x \neq 1$ we get x^2+x+1. At $x=0$ we have $\boxed{3}$.
b) The function contains the prototype $\sin(1/x)$ function, which has $\boxed{\text{no limit}}$ at $x=0$.
c) We had seen in class that $\lim_{x \to 0} 1/\log|x| = 0$ because $\log|x| \to -\infty$. Therefore $\sin(1/\log|x|) \to 0$. Assigning the value $f(0)=0$ makes the function continuous.
d) The function can is hopelessly discontinuous at $x=0$. For $|x| \to 0$ we have $\sin(x) \to 0$ and $\log|\sin(x)| \to -\infty$.
e) We can write this as $1-1/x$. This is a prototype case $1/x$ where the function converges to ∞.

Problem 5) Chain rule (10 points)

a) (2 points) Write $1+\cot^2(x)$ as an expression which only involves the function $\sin(x)$.

b) (3 points) Find the derivative of the function $\text{arccot}(x)$ by using the chain rule for

$$\cot(\text{arccot}(x)) = x \; .$$

c) (2 points) Write $1+\tan^2(x)$ as an expression which only involves the function $\cos(x)$.

d) (3 points) Find the derivative of the function $\arctan(x)$ by using the chain rule for

$$\tan(\arctan(x)) = x \; .$$

Remark: even if you should know the derivatives of arccot or arctan, we want to see the derivations in b) and d).

Solution:
We have done a),b) in homework problem 3) of Lecture 10.
a) $1+\cot^2(x) = 1+\cos^2(x)/\sin^2(x) = (\sin^2(x)+\cos^2(x))/\sin^2(x) = 1/\sin^2(x)$.
b) $\cot'(x) = -1/\sin^2(x) = 1+\cot^2(x)$ implies $(1+\cot^2(\text{arccot}(x)))\text{arccot}'(x) = 1$ and so $\text{arccot}'(x) = -1/(1+x^2)$.
We have done c),d) in class (on the side blackboard).
c) $1+\tan^2(x) = 1+\sin^2(x)/\cos^2(x) = 1/\cos^2(x)$.
d) $d/dx \cot(\arctan(x)) = (1/\cos^2(\arctan(x)))\arctan'(x) = (1+\tan^2(\arctan(x)))\arctan'(x) = 1$. Therefore, $\arctan'(x) = 1/(1+x^2)$.
We will come back to this later in the course.

Problem 6) Derivatives (10 points)

Find the derivatives of the following functions:

a) (2 points) $f(x) = \frac{\cos(3x)}{\cos(x)}$

b) (2 points) $f(x) = \sin^2(x)\log(1+x^2)$

c) (2 points) $f(x) = 5x^4 - \frac{1}{x^2+1}$

d) (2 points) $f(x) = \tan(x) + \exp(-\sin(x^2))$

e) (2 points) $f(x) = \frac{x^3}{(1+x^2)}$

Solution:
a) Use the quotient rule $[-3\sin(3x)\cos(x) + \sin(x)\cos(3x)]/\cos^2(x)$.
b) Use the chain rule and the product rule $2\sin(x)\cos(x)\log(1+x^2) + 2x\sin^2(x)/(1+x^2)$.
c) Use the quotient and chain rule for the second summand $20x^3 + (2x)/(x^2+1)^2$.
d) The second sum uses the chain rule twice $1/\cos^2(x) + e^{-\sin(x^2)}(-\cos(x^2))2x$.
e) Use the quotient rule $(3x^2(1+x^2) - x^3(2x))/(1+x^2)^2$.

Problem 7) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ for the following functions f at $x = 0$ or state (providing reasoning as usual) that the limit does not exist.

a) (2 points) $f(x) = \frac{\sin(3x)}{\sin(x)}$

b) (2 points) $f(x) = \frac{\sin^2(x)}{x^2}$

c) (2 points) $f(x) = \sin(\log(|x|))$

d) (2 points) $f(x) = \tan(x)\log(x)$

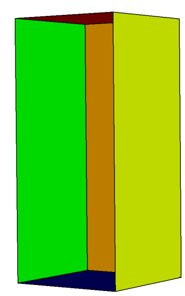
e) (2 points) $f(x) = \frac{(5x^4-1)}{(x^2+1)}$

Solution:
a) Apply l'Hospital once, to get the limit $\boxed{3}$.
b) $\boxed{1}$ since it is the square of $\sin(x)/x$ which has limit 1. One could also use l'Hospital twice.
c) There is $\boxed{\text{no limit}}$ because $log(|x|)$ goes to $-\infty$ and $\sin(\log|x|)$ oscillates indefinitely.
d) $\boxed{0}$ as we have done in class. Write as $[\sin(x)\log(x)]\cos(x)$ and $\cos(x)$ has no problem at $x = 0$. The limit $\sin(x)\log(x)$ is the same as $x\log(x)$ which we have done in class.
e) $\boxed{-1}$. There is no problem at this point because the nominator is not zero. We can just plug in $x = 0$ and get the value.

Problem 8) Extrema (10 points)

A rectangular shoe-box of width x, length x and height y is of volume 2 so that $x^2y = 2$. The surface area adds up three rectangular parts of size $(x \times y)$ and 2 square parts of size $(x \times x)$ and leads to

$$f = 2x^2 + 3xy \ .$$



a) (2 points) Write down the function $f(x)$ of the single variable x you want to minimize.

b) (6 points) Find the value of x for which the surface area is minimal.

c) (2 points) Check with the second derivative test, whether the point you found is a local minimum.

Solution:
a) Solve for $y = 2/x^2$ and substitute it into f. The function is $f(x) = 2x^2 + 6/x$.
b) $f'(x) = 4x - 6/x^2 = 0$ for $2x^3 = 3$ so that $x = (3/2)^{(1/3)}$.
c) $f''(x) = 4 + 12/x^3$. The second derivative is positive at the critical point. The critical point is a local minimum.

Problem 9) Global extrema (10 points)

In this problem we study the function $f(x) = 3x^5 - 5x^3$ on the interval $[-2, 2]$.

a) (2 points) Find all roots of f.
b) (2 points) Find all local extrema of the function.
c) (2 points) Use the second derivative test to analyze the critical points, where appli-

cable.
d) (2 points) Find the **global** maximum and minimum of f on the interval $[-2, 2]$.
e) (2 points) Bring the function $x \log(x)$ into indefinite form at 0.

Solution:
a) The roots are $\boxed{0, 0, 0, -\sqrt{5/3}, \sqrt{5/3}}$ as you can see by factoring x^3 out. The root 0 is a triple root.
b) The derivative is $15x^4 - 15x^2 = 15x^2(x^2 - 1)$ which has roots at $\boxed{0 \text{ and } 1 \text{ and } -1}$. These are candidates for local extrema.
c) The second derivative is $30x(2x^2 - 1)$. At $x = 0$, the second derivative is zero. The second derivative test does not apply at this point. At $x = 1$, the second derivative is positive at $x = -1$ it is negative. $\boxed{x = 1 \text{ is a local min, and } x = -1 \text{ is a local max.}}$
d) We compare $(f(2), f(-2), f(1), f(-1), f(0)) = (56, -56, -2, 2, 0)$ to see that the global extrema are located at the boundary. The point $\boxed{2}$ is the global maximum and the point $\boxed{-2}$ is the global minimum.
e) $\log(x)/(1/x)$.