Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

3/5/2021: First hourly Practice C

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 6 AM on March 6th.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F — If f is concave up on $[0, 1]$ and concave down on $[1, 2]$ then 1 is an inflection points.

Solution:
Indeed, f'' changes sign there.

2) T F — The function $f(x) = \exp(x)$ has the root $x = 1$.

Solution:
exp does not have any root.

3) T F — $\log(\exp(1)) = 1$, if log is the natural log and $\exp(x) = e^x$ is the exponential function.

Solution:
Yes, exp is the inverse of log.

4) T F — The chain rule assures that $d/dx f(f(x)) = f'(f(x))f'(x)$.

Solution:
Yes, this is a special case

5) T F — The function $x^2/(1 + x^2)$ is continuous everywhere on the real axes.

Solution:
The denominator is never zero so that there is no pole.

6) T F — The function $\cot(x)$ is the inverse of the function $\tan(x)$.

Solution:
No, it is $\text{arccot}(x)$ which is the inverse.

7) T **F** The function $\tan(x)/\log|x|$ defines an indefinite form at 0.

Solution:
No, it is of the form $0/\infty$.

8) T **F** $\cos(\pi/2) = 1/2$.

Solution:
The cosine has a root at $x = \pi/2$.

9) T **F** If a function f is differentiable on $[-1, 1]$, then there is a point x in that interval where $f'(x) = 0$.

Solution:
take $f(x) = x$.

10) **T** F The chain rule assures that $d/dx(g(x^2)) = 2xg'(x^2)$.

Solution:
Yup. This is the formula!

11) **T** F We have $\lim_{x\to\infty}((x^2 + 1)/x^2) = 1$

Solution:
This is a consequence of l'Hospital's rule when applied twice.

12) T **F** An inflection point is a point, where the function $f'(x)$ changes sign.

Solution:
It is a point, where the second derivative changes sign

13) T F

If $f''(-2) > 0$ then f is concave up at $x = -2$.

Solution:
The slope of the tangent increases which produces a concave up graph. One can define concave up with the property $f''(x) > 0$

14) T F

The intermediate value theorem assures that the continuous function $x + \sin(x) = 0$ has a root.

Solution:
The intermediate value theorem deals with roots.

15) T F

We can find a value b and define $f(0) = b$ such that the function $f(x) = (x^{28} - 1)/(x^2 - 1)$ is continuous everywhere.

Solution:
We divide by zero at $z = 1$.

16) T F

If the third derivative $f'''(x)$ is negative and $f''(x) = 0$ then f has a local maximum at x.

Solution:
It is the second derivative test, not the third one

17) T F

If $f(x) = x^2$ then $Df(x) = f(x + 1) - f(x)$ has a graph which is a line.

Solution:
$Df(x) = 2x - 1$.

18) T F

The quotient rule is $d/dx(f/g) = f'(x)/g'(x)$.

Solution:
No, it is not related to the l'Hospital rule.

19) T F With $Df(x) = f(x+1) - f(x)$, we have $D(1+a)^x = a(1+a)^x$.

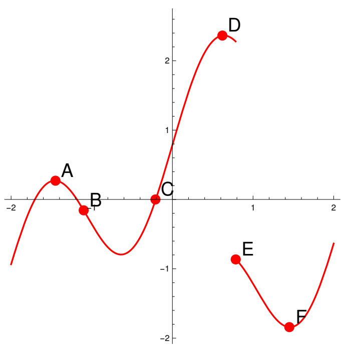
Solution:
Compound interest

20) T F It is true that $\log(5)e^x = e^{x\log(5)}$ if $\log(x)$ is the natural log.

Solution:
This is no identity.

Problem 2) Matching problem (10 points) No justifications are needed.

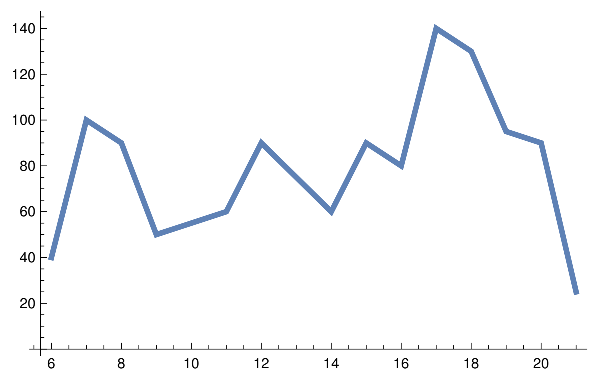
a) (6 points) You see the graph of a function $f(x)$ defined on $[-2, 3]$. Various points $(x, f(x))$ are marked. Match them:



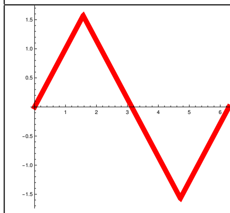
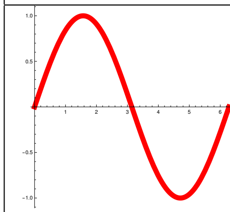
Point x is	Fill in A-F
Local maximum	
Root	
Inflection point	
Discontinuity	
Global maximum	
Local minimum	

b) (2 points) the **Harvard recreation** publishes regularly a graph of a function $f(x)$ which shows the number of people at the **Mac** gym as a function of time. At 5 o'clock, there are in average 120 visitors, at 9 in the morning, there are 60 people working out. By the intermediate value theorem, there must be a moment at which exactly $\pi^4 = 97.5...$ visitors are present. This is obviously nonsense. Where is the flaw?

Reason	Check one
No differentiability	
Statistical glitch	
No Continuity	
Inaccurate data	



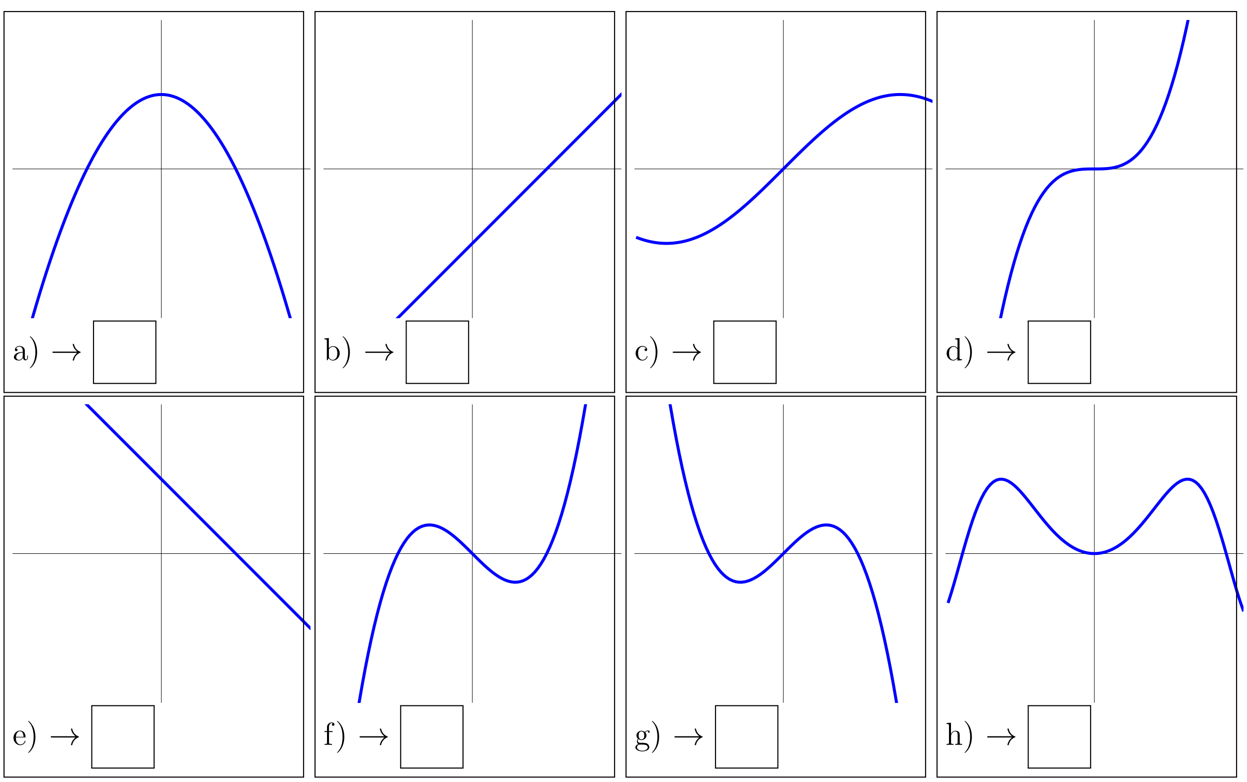
c) (2 points) In front of the **"Class of 1959 Chapel"** at the Harvard business school is an amazing clock: a marble tower contains a steel pole and a large bronze ball which moves up and down the pole indicating the time of the day. As the ball moves up and down the pole, lines with equal distance on the tower indicate the time. At noon, the sphere is at the highest point. At midnight it is at the bottom. It moves the same distance in each hour. If we plot the hight of the sphere as a function of time, which graph do we see?

The height function	Check which applies
	
	

Solution:
a) ACBEDF
b) Lack of continuity. The number of people is an integer.
c) It is the piecewise linear function because the clock moves with constant speed.

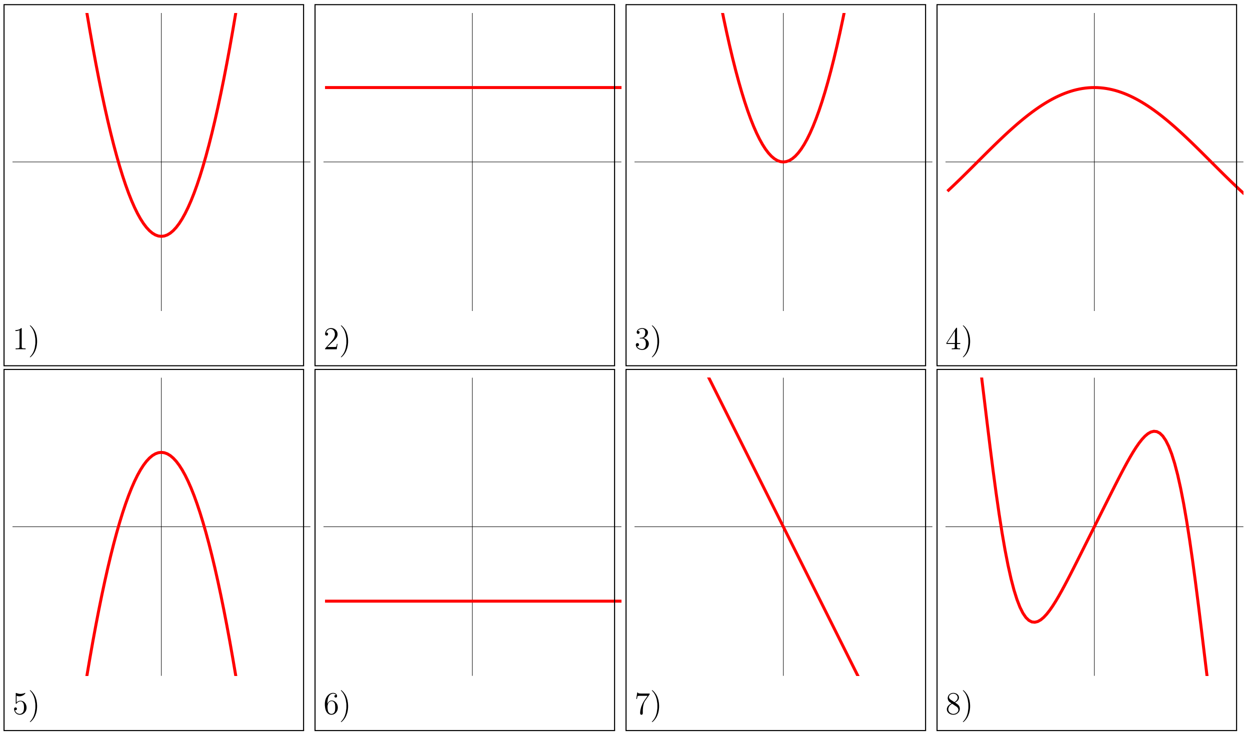
Problem 3) Matching problem (10 points) No justifications are needed.

Match the graph of the functions f in $a) - h)$ with the derivatives f' in 1)-8).



a) $\rightarrow$ ☐ b) $\rightarrow$ ☐ c) $\rightarrow$ ☐ d) $\rightarrow$ ☐

e) $\rightarrow$ ☐ f) $\rightarrow$ ☐ g) $\rightarrow$ ☐ h) $\rightarrow$ ☐



Solution:
7243, 6158

Problem 4) Continuity (10 points)

Each of the following functions has a point x_0, where the function is not defined. Find the limit $\lim_{x\to x_0} f(x)$ or state that the limit does not exist.

a) (2 points) $f(x) = \frac{x^3-8}{x-2}$, at $x_0 = 2$

b) (2 points) $f(x) = \sin(\sin(\frac{1}{x})) - \tan(x)$, at $x_0 = 0$

c) (2 points) $f(x) = \frac{\cos(x)-1}{x^2}$, at $x_0 = 0$

d) (2 points) $f(x) = \frac{\exp(x)-1}{\exp(5x)-1}$, at $x_0 = 0$

e) (2 points) $f(x) = \frac{x-1}{x}$, at $x_0 = 0$

Solution:
a) We can use Hospital's rule to see that the limit is $\lim_{x \to 2} 3x^2/1 = \boxed{12}$.
b) There is no way that we can save the oscillatory singularity. $\boxed{\text{No limit exists}}$.
c) Apply Hospital twice to see that the limit is $\boxed{-1/2}$.
d) Apply l'Hospital to see that the limit is $\lim_{x \to 0} e^x/(5e^{5x}) = \boxed{1/5}$.
e) This can not be saved at $x = 0$. There is a pole there. $\boxed{\text{No limit exists.}}$

Problem 5) Derivatives (10 points)

Find the derivatives of the following functions. If you use a differentiation rule, note which one you use.

a) (2 points) $f(x) = \sin(7x) + (1 + x^2)$.

b) (2 points) $f(x) = \frac{\sin(7x)}{(1+x^2)}$.

c) (2 points) $f(x) = \sin(7 + x^2)$.

d) (2 points) $f(x) = \sin(7x)(1 + x^2)$.

e) (2 points) $f(x) = \sin(7x)^{(1+x^2)}$

Solution:
a) Chain rule $7\cos(7x) + 2x$.
b) Quotient rule $[(1 + x^2)7\cos(7x) - \sin(7x)2x]/(1 + x^2)^2$.
c) Chain rule. $2x\cos(7 + x^2)$.
d) Product and chain rule. $7\cos(7x)(1 + x^2) + 2x\sin(7x)$.
e) First write as $\exp((1 + x^2)\log(\sin(7x)))$ and differentiate now with the chain rule: $\exp((1 + x^2)\log(\sin(7x)))[2x\log(\sin(7x)) + (1 + x^2)7\cos(7x)/\sin(7x)]$.

Problem 6) Limits (10 points)

Find the limits $\lim_{x \to 0} f(x)$ for the following functions f at $x = 0$:

a) (2 points) $f(x) = \frac{(1-\exp(11x))}{(1-\exp(3x))}$

b) (2 points) $f(x) = \frac{\sin(\sin(5x))}{\sin(7x)}$

c) (2 points) $f(x) = \frac{\log(x)}{\log(5x)}$

d) (2 points) $f(x) = \frac{x^2 \cos(x)}{\sin^2(x)}$

e) (2 points) $f(x) = \frac{(1+1/x^2)}{(1-1/x^2)}$

Solution:
All with Hospital.
a) 11/3
b) 5/7
c) 1
d) 1
e) -1

Problem 7) Trig functions (10 points)

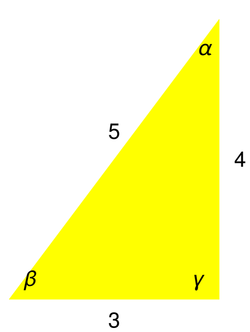
A triangle with side lengths $3, 4, 5$ has a right angle. Let $\alpha < \beta < \gamma$ denote the angles ordered by size.

a) (4 points) What are the numerical values of $\cos(\alpha), \cos(\beta), \cos(\gamma), \sin(\gamma)$?

b) (2 points) Find the numerical value of $\tan(\alpha)$ and $\cot(\alpha)$.

The next problem is independent of the previous two.

c) (4 points) Find the derivative of the inverse function of $\arcsin(x)$ by starting with the identity $x = \sin(\arcsin(x))$. Your derivation of $\arcsin'(x)$ should convince somebody who does not know the identity already.



Solution:
a) $\cos(\alpha) = 4/5, \cos(\beta) = 3/5 \cos(\gamma) = 0, \sin(\gamma) = 1$ b) $3/4, 4/3$
c) Differentiating $x = \sin(\arcsin(x))$ gives $1 = \cos(\arcsin(x)) \arcsin'(x)$ solve for $\arcsin'(x)$ and replace $\cos(u)$ with $\sqrt{1 - \sin^2(u)}$ to get $\arcsin'(x) = 1/\sqrt{1 - x^2}$.

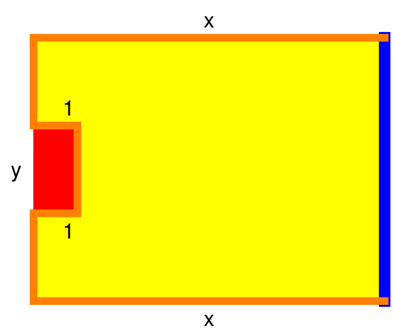
Problem 8) Extrema (10 points)

A tennis field of width x and length y contains a fenced referee area of length 2 and width 1 within the field and an already built wall. The circumference a fence satisfies $2x + y + 2 = 10$, (an expression which still can be simplified). We want to maximize the area $xy - 2$.

a) (2 points) On which interval $[a, b]$ does the variable x make sense? Find a function $f(x)$ which needs to be maximized.

b) (6 points) Find the local maximum of x and check it with the second derivative test.

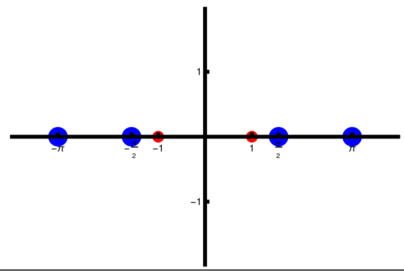
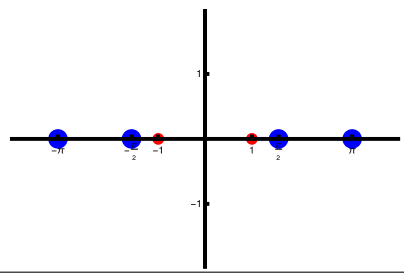
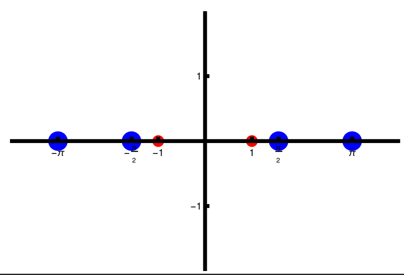
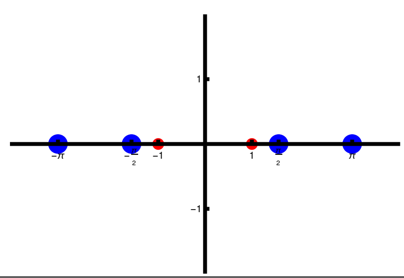
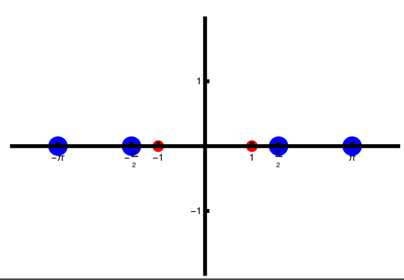
c) (2 points) What is the global maximum of f on $[a, b]$?



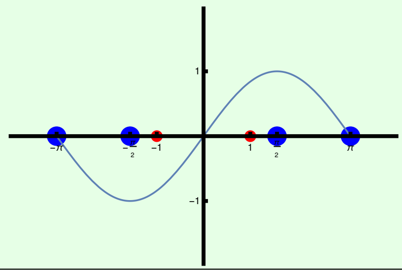
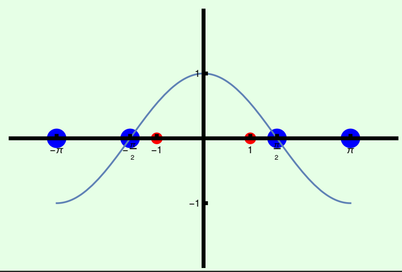
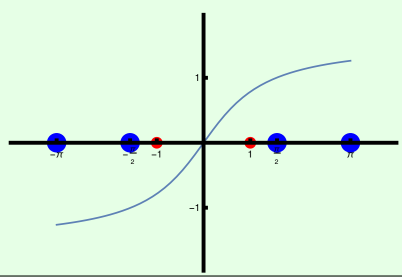
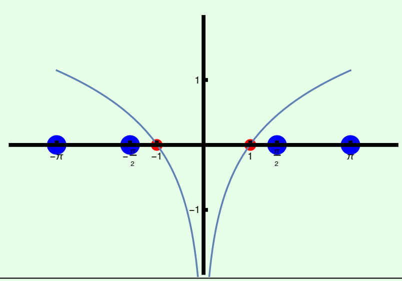
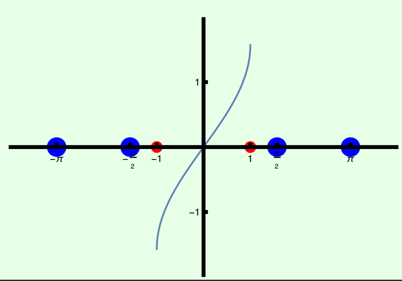
Solution:
a) We must have $x \geq 0$ and $y = 8 - 2x > 0$ so that $x \leq 4$ The interval on which f is defined is therefore $[0, 4]$. Actually, since $y \geq 2$ we must even have $x \in [0, 3]$ but both bounds were considered correct. Since the area has to be positive, one could even shave off a bit more from the lower bound but that was not necessary.
b) The function to extremize is $f(x) = x(8 - 2x) - 2$. Its derivative is $8 - 4x$. This is zero at $x = 2$. We have $y = 4$. The maximal area is $xy - 2 = 8 - 2 = 6$.
c) Compare $f(0) = -2, f(4) = -2$ and $f(2) = 6$ to see that 6 is the maximal value.

Problem 9) Trig functions (10 points)

a) In the following five problems, find the numerical value and then draw the graph of the function.

Problem	Answer	Graph
A) (2 points) What is $\sin(\pi/3)$? Plot $\sin(x)$.		
B) (2 points) What is $\cos(5\pi/2)$? Plot $\cos(x)$.		
C) (2 points) Find $\arctan(1)$ Plot $\arctan(x)$.		
D) (2 points) What is $\log(1)$ Plot $\log\|x\|$.		
E) (2 points) What is $\arcsin(\sqrt{3}/2)$. Plot $\arcsin(x)$		

Solution:
A) $\sqrt{3}/2$.
B) 0
C) $\pi/4$
D) 0
E) $\pi/3$ The functions needed to be plotted with roots at the correct place.

Problem	Answer	Graph
a) (2 points) What is $\sin(\pi/3)$? Plot $\sin(x)$.		
b) (2 points) What is $\cos(5\pi/2)$? Plot $\cos(x)$.		
c) (2 points) Find $\arctan(1)$ Plot $\arctan(x)$.		
d) (2 points) What is $\log(1)$ Plot $\log\|x\|$.		
e) (2 points) What is $\arcsin(\sqrt{3}/2)$. Plot $\arcsin(x)$		

b) Simplify the following terms. log denotes the natural log and $\log_{10}$ denotes the log to the base 10. All answers are integers.

A) (2 points) $\exp(\log(2))$
B) (2 points) $e^{\log(2)3}$
C) (2 points) $\log(\log(e))$
D) (2 points) $\exp(\log(2) + \log(3))$
E) (2 points) $\log_{10}(10000)$

Solution:
a) 2
b) 8
c) 0
d) 6
e) 4