Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

3/5/2021: First hourly Practice A

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as PDF handwritten in a file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 6 AM on March 6th.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) T F The function $f(x) = \log(x)$ has a root at $x = 1$.

Solution:
Yes, log(1)=0.

2) T F $f(x) = x^2 + \exp(-x^2)$ has a root on the real axes.

Solution:
No, it is always positive

3) T F $\cos(\pi) = -1$.

Solution:
You know that.

4) T F $f(x) = \sin(|x|)$ is differentiable everywhere.

Solution:
It is not at $x = 0$ or $x = k\pi$ with integer k.

5) T F The chain rule assures that $\frac{d}{dx} \sin(x^5) = \cos(x) 5x^4$.

Solution:
This is not the correct application.

6) T F The function $f(x) = \exp(-\sin(x))$ is continuous everywhere.

Solution:
It is a combination of continuous functions

7) T **F** $\sinh(x) = (e^x - e^{-x})/2$ is positive everywhere.

Solution:
It is zero at $x = 0$ and negative for $x < 0$.

8) T **F** $\cot(x)$ is the inverse function of $\tan(x)$.

Solution:
It is the arctan, not the cotan

9) **T** F If $f(x)$ is differentiable at 0, then $f(x)^2$ is differentiable at 0.

Solution:
Differentiate to get $2f(x)f'(x)$.

10) **T** F The function $x^3/(1 + x^3)$ defines an indefinite form at $x = \infty$.

Solution:
Yes, it is of the form ∞/∞.

11) T **F** $\sin(3\pi/4) = -1$.

Solution:
It is positive and $\sqrt{1/2}$

12) T **F** $f(x) = \tan(x)$ has a vertical asymptote at $x = 0$.

Solution:
The cot function has this property.

13) **T** F e^{x^2} takes the value π at some point.

Solution:
Yes, by the intermediate value theorem, there is a place, where the value is 1 and places where f is arbitrary large

14) **T** F If $f(x) = x^2$, then $Df(x) = f(x+1) - f(x) = 2x + 1$.

Solution:
Yes, one check this formally $(x+1)^2 - x^2 = 2x + 1$.

15) T **F** The intermediate value theorem implies Rolles theorem.

Solution:
It is the mean value theorem which implies this.

16) T **F** The function $\sin(1/x)$ is continuous everywhere.

Solution:
It is a basic oscillatory discontinuity.

17) **T** F $\frac{d}{dx} \arctan(x) = 1/(1 + x^2)$.

Solution:
An important one.

18) T F $\frac{d}{dx} \log(3+x) = 3/(3+x)$.

Solution:
It is $1/(3+x)$.

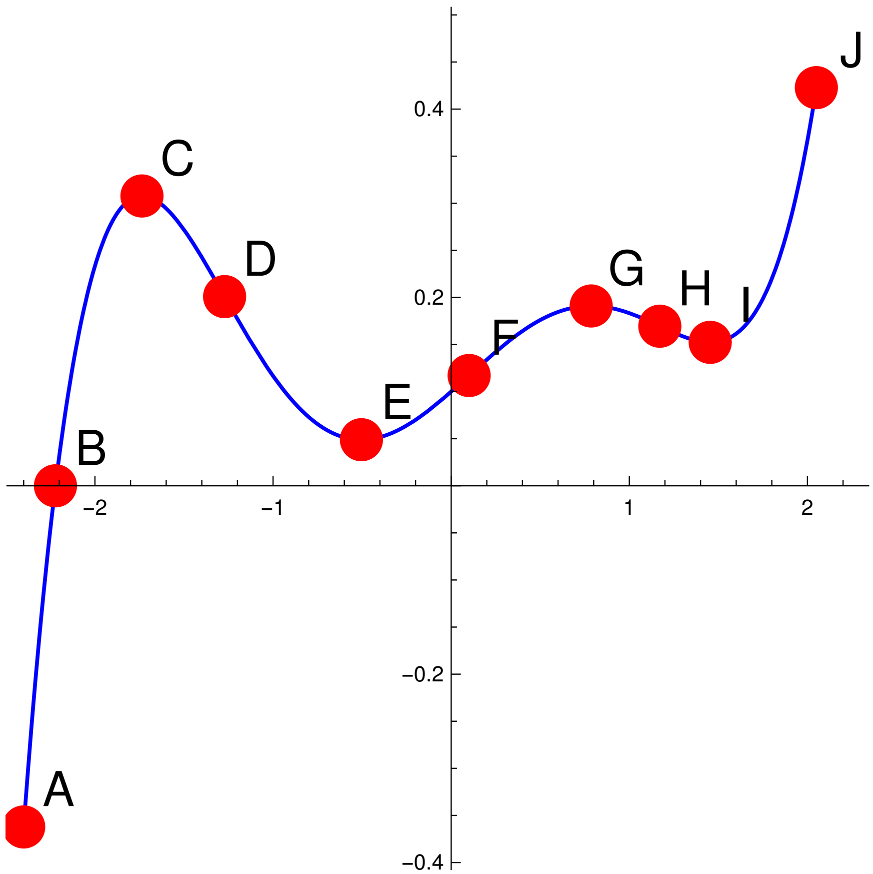
19) T F A continuous function on $[0,1)$ has at least one maximum.

Solution:
The interval is not closed. Bolzano does not apply.

20) T F The derivative of f/g is $(fg' - f'g)/g^2$.

Solution:
It is low D high minus high d low!

Problem 2) Choice problem (10 points) No justifications are needed.



a) (2 points) List the points A-J which are roots of f.

b) (2 points) List the points A-J are inflection points.

c) (2 points) List the points A-J that are local maxima.

d) (2 points) List the points A-J that are local minima.

e) (2 points) List the points A-J that are global maxima.

Solution:

a) B

b) D, F, H.

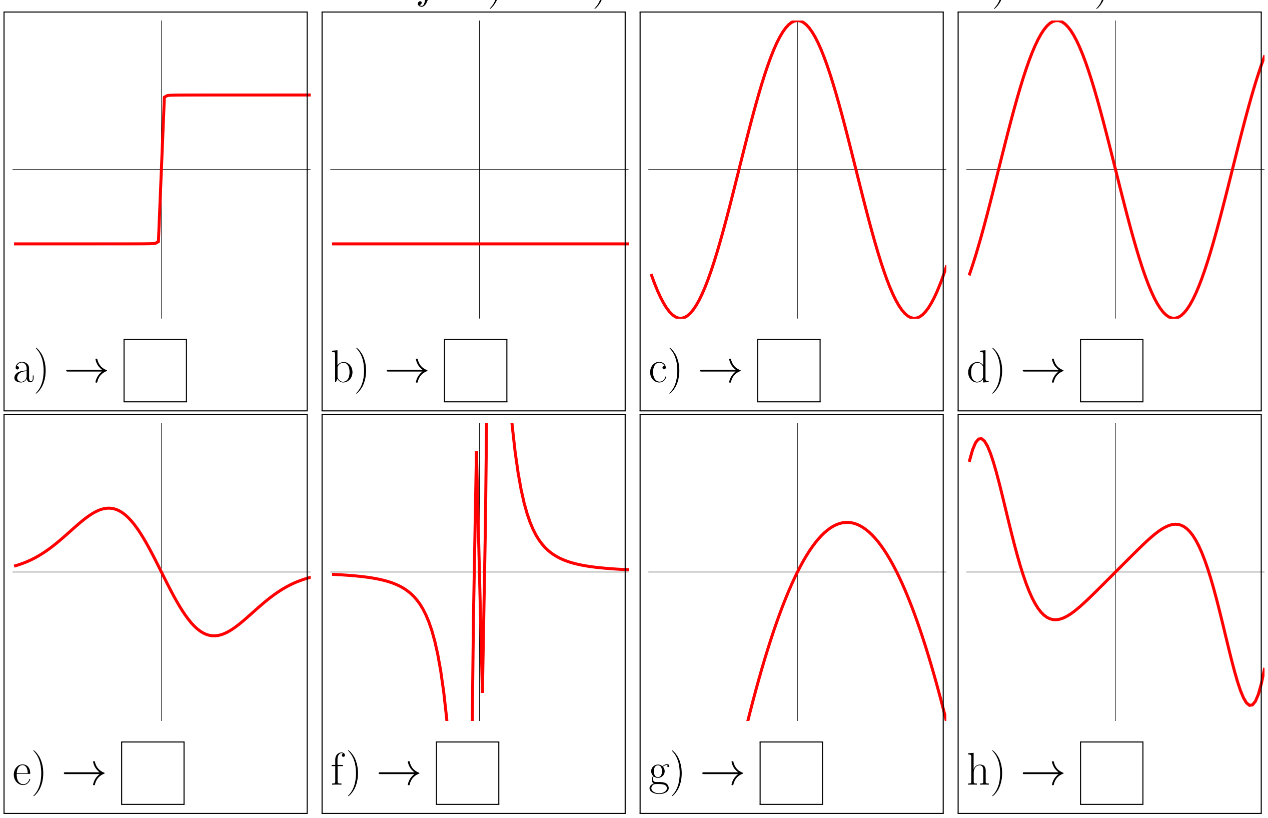
c) C, G

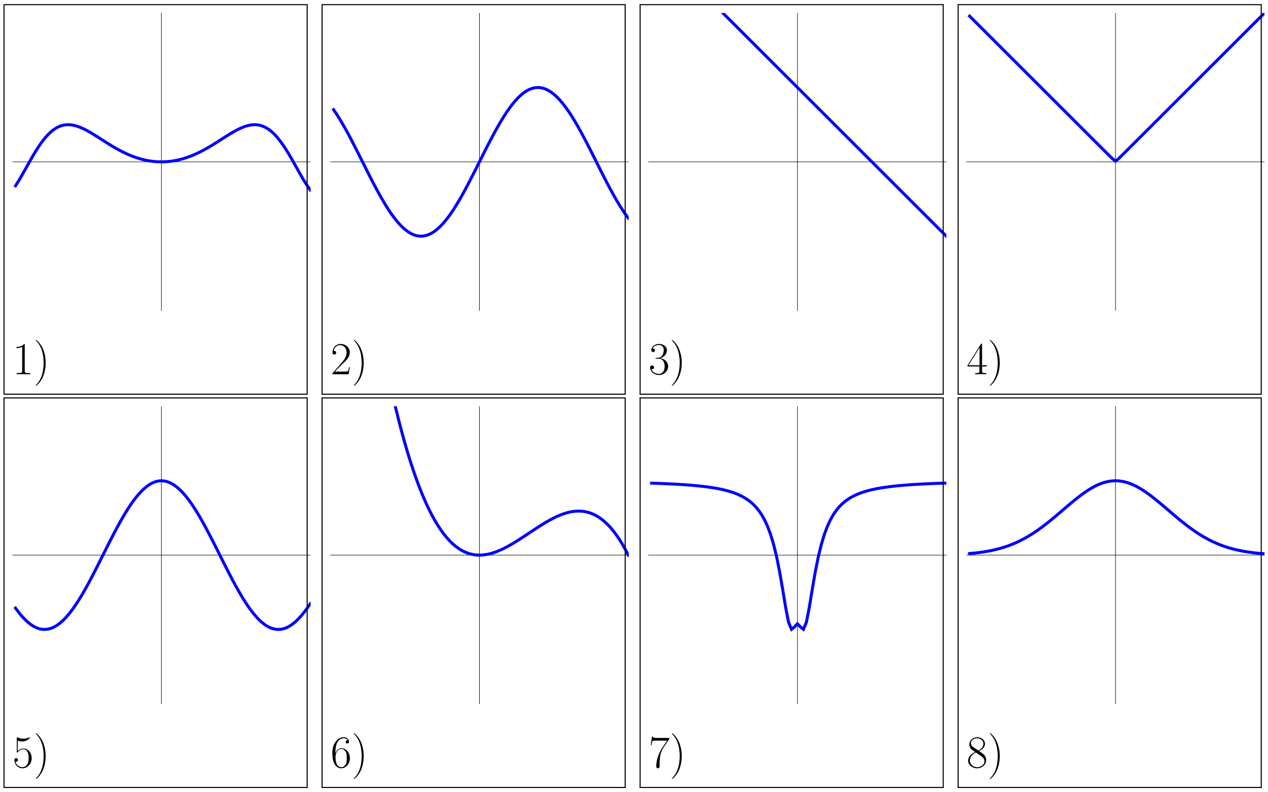
d) E, I

e) J.

Problem 3) Matching problem (10 points) No justifications are needed.

Match the derivatives f' a) to h) with the functions 1) to 8).





Solution:

4325

8761

Problem 4) Continuity (10 points)

Which of the following functions are continuous on $[-1, 1]$? As usual we extend the domain of definition to points, where a continuation is possible. In each case make the decision "continuous" or "not continuous" and point to the x value which needs special attention.

a) (2 points) $f(x) = \frac{x^6-1}{x^2-1}$

b) (2 points) $f(x) = \frac{\sin(\sin(x))}{\sin(\sin(\sin(x)))}$.

c) (2 points) $f(x) = \frac{\sin^2(x)}{2+\sin(x^2)}$

d) (2 points) $f(x) = \log|x|e^x$

e) (2 points) $f(x) = \frac{\sin(\tan(x))}{\sin(x)}$

Solution:

a) Continuous. There is only one point which needs to be checked: x=1 Hospital gives the limit $6/2 = 3$ at $x = 1$.

b) Continuous. There is only point 0 which needs to be considered. The limit at $x = 0$ is 1.

c) Continuous. The denominator is always positive. The function is continuous.

d) Not continuous. There is no way to save this at $x = 0$ as $\log|x|$ goes to $-\infty$ and $e^0 = 1$.

e) Continuous. We can use Hospital to get the value 1 at $x = 0$ or $x = \pi$ etc.

Problem 5) Derivatives (10 points)

Find the derivatives of the following functions. In each case, indicate which differentiation rule you use.

a) (2 points) $f(x) = \frac{1}{1+e^x}$.

b) (2 points) $f(x) = \cos(x)\sin(x)$.

c) (2 points) $f(x) = \frac{1+x^3}{1+x^2}$.

d) (2 points) $f(x) = \arctan(\sin(x))$.

e) (2 points) $f(x) = \log(\log(5x))$.

Solution:
a) $e^x/(1+e^x)$.
b) $\cos^2(x) - \sin^2(x)$.
c) $[3x^2(1+x^2) - (1+x^3)2x]/(1+x^2)^2$
d) $\cos(x)/(1+\sin^2(x))$. e) $1/(x\log(5x))$.

Problem 6) Limits (10 points)

Find the limits $\lim_{x \to 0} f(x)$ for the following functions:

a) (2 points) $f(x) = \frac{\exp(7x)-1}{\exp(4x)-1}$.

b) (2 points) $f(x) = \frac{x-1}{x+1}$.

c) (2 points) $f(x) = \frac{\arctan(x)}{\sin(x)}$.

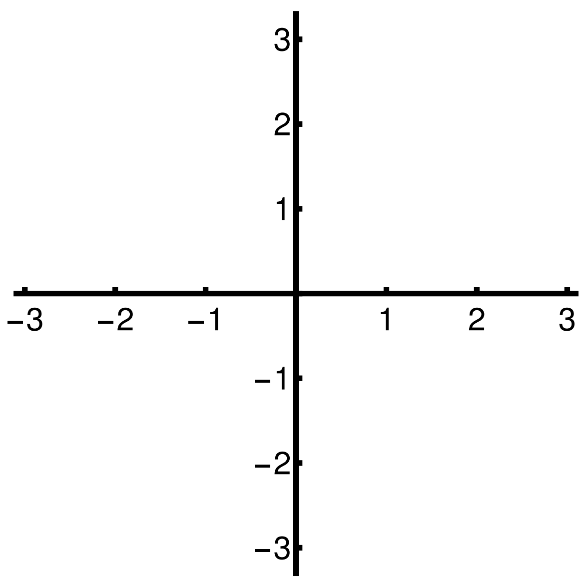
d) (2 points) $f(x) = \frac{\log(x^3)}{\log(x^2)}$.

e) (2 points) $f(x) = \frac{\sin(3x)\sin(5x)}{\sin(7x)\sin(2x)}$.

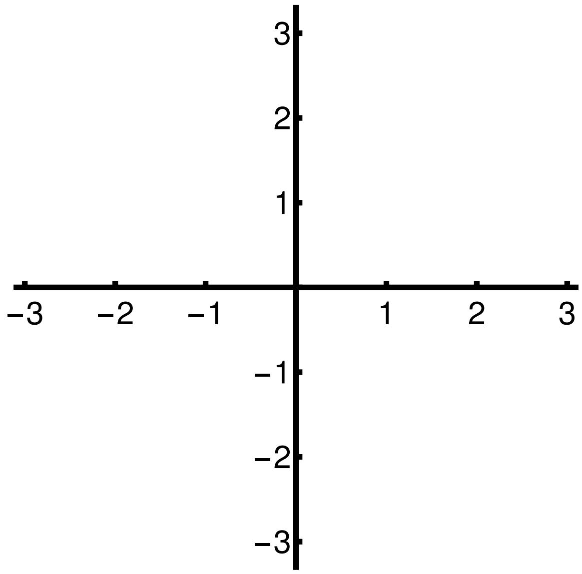
Solution:
All with Hospital.
a) 7/4.
b) -1 (no limit needs to be taken as the values are finite.
c) 1
d) 3/2
e) Take apart: $\sin(3x)/\sin(7x)$ goes to 3/7 and $\sin(5x)/\sin(2x)$ goes to 5/2. We have 15/14.

Problem 7) Functions (10 points)

a) (5 points) Draw the graph of the natural log function $f(x) = \log|x| = \ln|x|$. Make sure you get the roots and asymptotes correct.



b) (5 points) Draw the graph of the arctan function $f(x) = \arctan(x)$. Make sure you get the roots and asymptotes correct.



Problem 8) Extrema (10 points)

a) (3 points) Find all the critical points of the function

$$f(x) = x^3 - 3x + 1 \ .$$

b) (3 points) Use the second derivative test to classify the critical points of f.

c) (2 points) On the interval $[-3, 3]$, where it the global maximum, and where is the global minimum?

d) (2 points) Which theorem assures that there is a global maximum and a global minimum on $[-3, 3]$?

Solution:
a) We have $f'(x) = 3x^2 - 3$ which is zero at $x = 1$ or $x = -1$.
b) The second derivative is $6x$. So $x = 1$ is a minimum and $x = -1$ is a maximum.
c) The global minimum is $x = 3$ and the minimum is $x = -3$.
d) The Bolzano extremal value theorem.

Problem 9) Algebra rules (10 points, 2 points each)

a)	$(e^x)^y$	
b)	e^{x+y}	
c)	$\log(xy)$	
d)	$\frac{\tan(x)}{\sin(x)}$	
e)	$\frac{x^9}{x^3}$	

Choose from the following expressions.

- $1/\cos(x)$
- $\cos(x)/\sin^2(x)$.
- $e^x e^y$
- $\log(x+y)$
- $e^x + e^y$
- $e^{(xy)}$
- x^6
- $\log(x) - \log(y)$

- $e^{(x^y)}$
- $\log(x) + \log(y)$
- x^3
- $\cos(x)$

Solution:
a) e^{xy}
b) $e^x e^y$.
c) $\log(x) + \log(y)$ d) $1/\cos(x)$.
e) x^6.