3/5/2021: First hourly

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

- Solutions are submitted to knill@math.harvard.edu as a PDF, handwritten in one file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False ⋯ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids or online tools or external information are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 6 AM on March 6th. Do not communicate with anybody in the class during the exam period.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1) [T] F — The function $f(x) = \log(\exp(x))$ has a root at $x = 0$.

Solution:
Yes, $\log \exp(x)) = x$.

2) [T] F — The "today" function $f(x) = x^3 - x^5 + x^{2021}$ has a real root.

Solution:
Yes, by the intermediate value theorem.

3) [T] F — $\cot(\pi/4) = 1$.

Solution:
You know that.

4) [T] F — $f(x) = |2 + \sin(x)|$ is differentiable everywhere.

Solution:
The absolute value does not matter.

5) T [F] — The chain rule assures that $\frac{d}{dx} \sin(\sin(x)) = \cos(\cos(x)) \cos(x)$.

Solution:
This is not the correct application.

6) [T] F — $f(x) = 1 + \frac{\sin(x)^2}{x}$ with assumption $f(0) = 1$ is continuous.

Solution:
It is a combination of continuous functions $1 + \text{sinc}(x)\sin(x)$.

7) T (highlighted) F — A differentiable function satisfying $f(0) = 1, f(1) = 0$ must have $f'(x)$ be negative somewhere in $[0, 1]$.

Solution:
Otherwise, we only would go up or stay on the same height.

8) T F (highlighted) — $\sec(x)$ is the inverse function of $\cos(x)$.

Solution:
It is $1/\cos(x)$.

9) T (highlighted) F — If $f(x)$ is differentiable at 0, and $f(0) = 0$, then $f(f(x))$ is differentiable at 0.

Solution:
Differentiate to get $2f'(x)f'(f(x))$ which is $2f'(0)f'(0)$.

10) T F (highlighted) — $f(x) = x^5/(1 + x^5)$ defines an indeterminate form at $x = 0$.

Solution:
It is of form $0/1$ and so not an indeterminate form.

11) T (highlighted) F — $f(x) = (\sin(\pi/2+h)-\sin(\pi/2))/h < 0$ for all positive h smaller than $\pi/2$.

Solution:
Yes, even if in the limit we get zero

12) T F The function $f(x) = x^7$ has a critical point at $x = 0$.

Solution:
Yes, even so it is neither a max nor min.

13) T F If f has an inflection point at 0, then $g(x) = 1 - f(x)$ has an inflection point at 0.

Solution:
Yes, also $-f''(0)$ changes sign.

14) T F $f(x) = \cot(x^2)$ has a vertical asymptote at $x = 0$.

Solution:
Yes

15) T F $f(x) = \sin(x)$ takes the value $\pi/2$ at some point x.

Solution:
Sin is between -1 and 1

16) T F If $f(x) = x(x-1)$, then $Df(x) = f(x+1) - f(x) = 2x$.

Solution:
Yes, one check this formally $(x+1)^2 - x^2 = 2x + 1$.

17) T F The function $f(x) = \sin(x)\sin(1/x)$ with the understanding $f(0) = 0$ is continuous everywhere.

Solution:
It is a basic oscillatory discontinuity.

18) T F $\frac{d}{dx}\arctan(x^2) = 2x/(1+x^2)$.

Solution:
An important one.

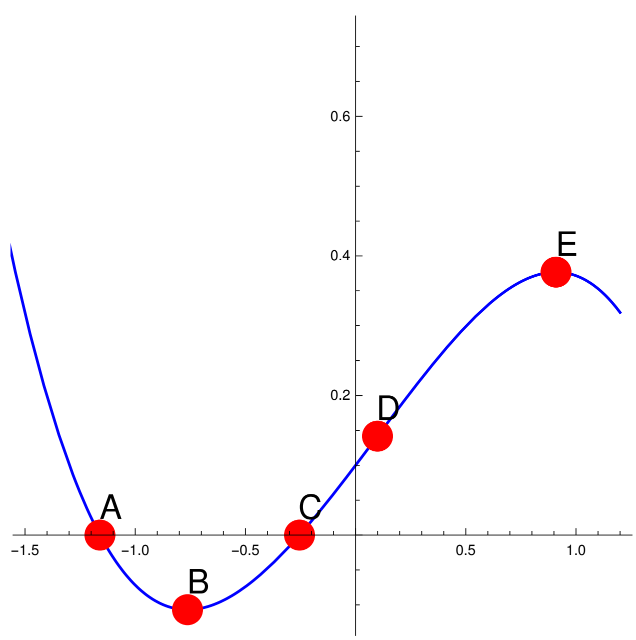
19) T F $\frac{d}{dx}\log(7-x) = -1/(7-x)$.

Solution:
Wrong sign.

20) T F The derivative of g/f is $(fg' - f'g)/f^2$.

Solution:
Yes, even so we have changed the letters

Problem 2) Choice problem (10 points) No justifications are needed.



a) (2 points) List the points A-E which are roots of f.

b) (2 points) List the points A-E that are critical points of f.

c) (2 points) List the points A-E that are critical points of f'.

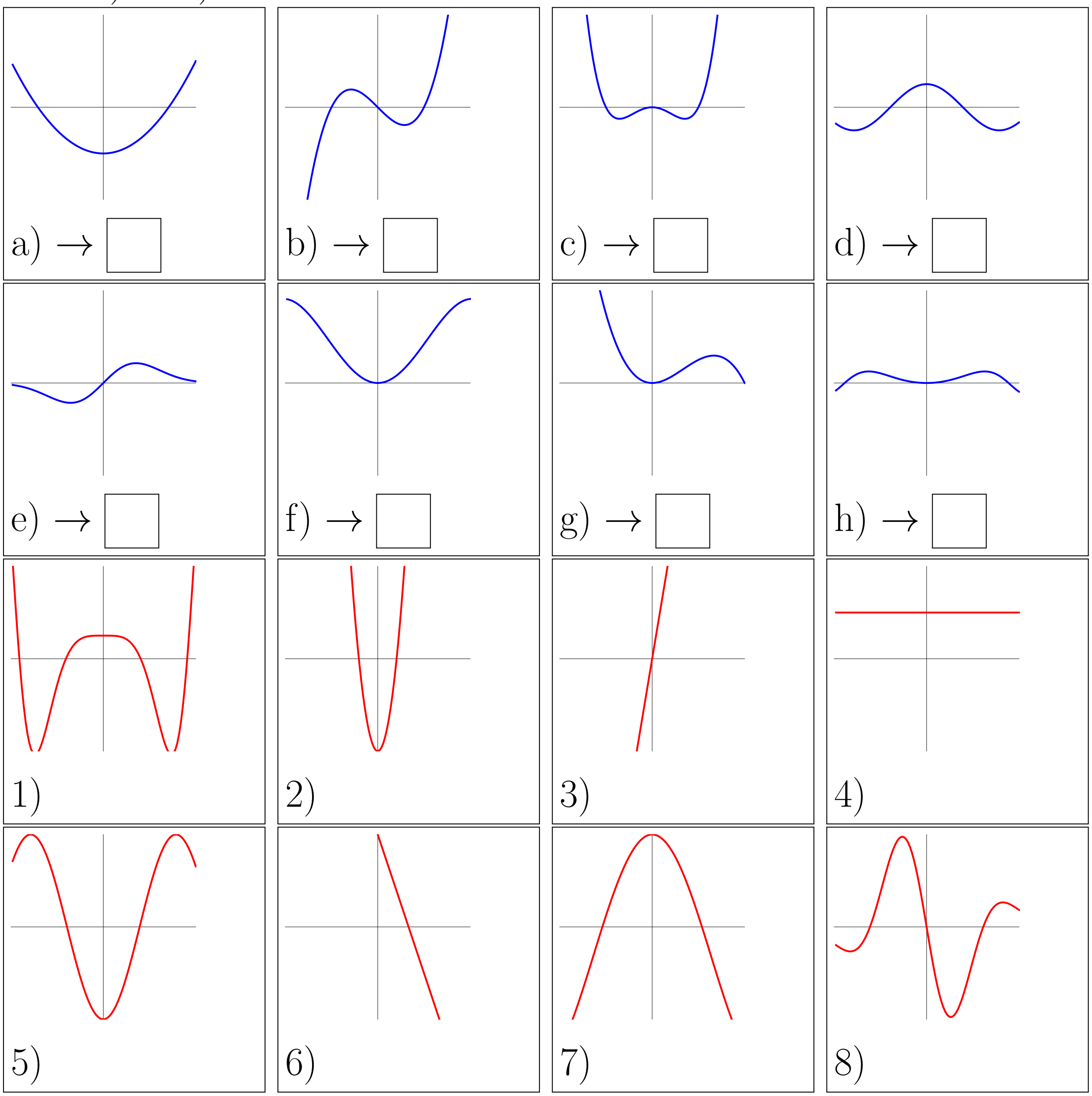
d) (2 points) List the critical points A-E that are local maxima.

e) (2 points) List the points A-E of f that are global minima on the given interval.

Solution:
a) AC, b) BE, c) D, d) E, e) B

Problem 3) Matching problem (10 points) No justifications are needed.

Match the functions f a) to h) with the second derivatives functions 1) to 8).



Solution:

4, 3, 2, 5, and 8, 7, 6, 1

Problem 4) Continuity (10 points)

Which of the following functions are continuous on $[-1, 1]$? As always, we extend continuity to functions for which a continuation is possible to initially not defined points, like $f(x) = (x^2 - 1)/(x - 1)$, which was considered continuous because with $f(1) = 2$, it becomes a continuous function. Make the decision "continuous" or "not continuous" in each of the cases a)-e) and point out any possible x values, which need special attention.

a) (2 points) $f(x) = \cos(\sin(5/x^2))$.

b) (2 points) $f(x) = x^2 \sin(5/x^2)$.

c) (2 points) $f(x) = \frac{\sin^2(x)}{x^2}$

d) (2 points) $f(x) = ||3x| - 4x|$

e) (2 points) $f(x) = 3x/|4x|$

Solution:

a) not continuous (oscillation)

b) continuous. We have $|f(x)| \leq |x|^2 \to 0$ for $x \to 0$.

c) continuous, fundamental theorem of trig

d) continuous, piecewise linear

e) not continuous (jump)

Problem 5) Derivatives (10 points)

Do the required computations for the following functions. In each case, indicate which differentiation rule you use.

a) (2 points) For $f(x) = \sin(x)e^x$, write down $T(x) = x - f(x)/f'(x)$.

b) (2 points) For $f(x) = \log(\sin(x))$, write down $f''(x)$.

c) (2 points) For $f(x) = e^{(e^{\sin(x)})}$, write down $f'(x)$.

d) (2 points) For $f(x) = x^x$, write down $f'(x)$.

e) (2 points) Demonstrate how one can obtain the derivative of $\arcsin(x)$ with the help of the chain rule.

Solution:
a) $x - \sin(x)/(\cos(x) + \sin(x))$.
b) $-sec^2(x)$
c) $e^{e^{\sin(x)}} e^{\sin(x)} \cos(x)$
d) $x^x(1 + \log(x))$.

Problem 6) Limits (10 points)

Find the limits $\lim_{x \to 0} f(x)$ for the following functions. As always give details and indicate what method you are using.

a) (2 points) $f(x) = \frac{1-\exp(4x)}{1-\exp(3x)}$.

b) (2 points) $f(x) = \frac{1-x^2}{\cos(3x)}$.

c) (2 points) $f(x) = \frac{3x}{\log(1+4x)}$.

d) (2 points) $f(x) = \frac{\log(x^3)}{\log(x^2)}$.

e) (2 points) $f(x) = x(\log(7x))^2$.

Solution:

a) 4/3 Hospital

b) 1 just plug in the number $x = 0$

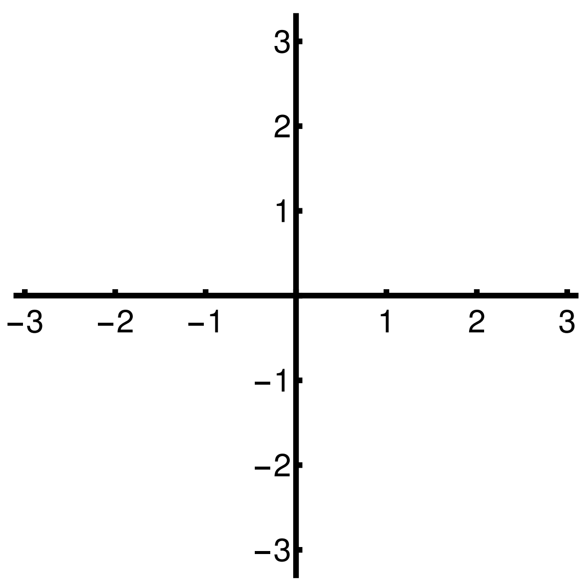
c) 3/4 Hospital

d) 3/2 Hospital and cancel

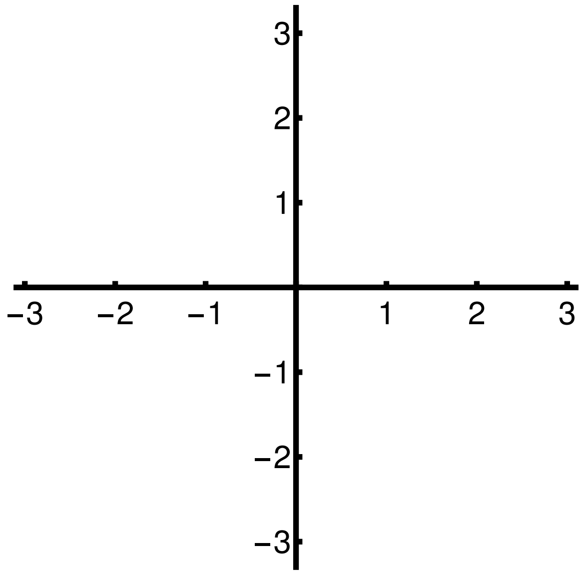
e) 0. This needs twice Hospital. First for $\log(7x)^2/(1/x)$ then again for $x\log(7x)$.

Problem 7) Functions (10 points)

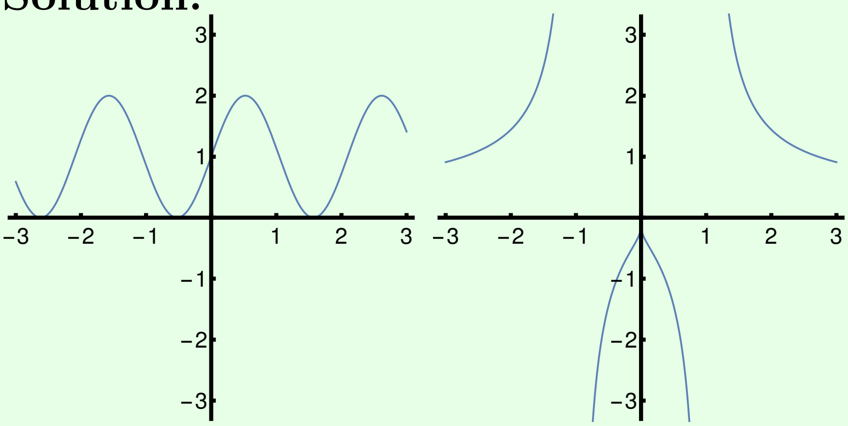
a) (5 points) Draw the graph of the function $f(x) = 1 + \sin(3x)$ on $[\pi, \pi]$. Mark the roots, the maxima, minima and inflection points. On paper, please copy the template below exactly!



b) (5 points) Draw the graph of $f(x) = 1/\log|x|$ on $[-3, 3]$. Make sure to indicate what the values at $x = -1, x = 0, x = 1$ are, and draw any possible asymptotes.



Solution:



Problem 8) Extrema (10 points)

a) (3 points) Find all the critical points of the function

$$f(x) = x^5 - 5x + 7 .$$

b) (3 points) Classify the critical points using the second derivative test.

c) (2 points) Find the global maximum and global minimum on $[-2, 2]$.

d) (1 point) Which mathematician proved that a global maximum and a global minimum exists for a continuous function on $[-2, 2]$?

e) (1 point) What is the name of that theorem on global maxima or minima?

Solution:
a) We have $f'(x) = 5x^4 - 5$ which is zero at $x = 1$ or $x = -1$.
b) The second derivative is $20x^3$. So $x = 1$ is a minimum and $x = -1$ is a maximum.
c) The global minimum is $x = 2$ and the minimum is $x = -2$.
d) Bolzano.
e) Extremal value theorem.

Problem 9) Algebra (10 points, 2 points each)

Simplify the following expressions. For example, $3^x 3^y$ can be written fewer letters as 3^{x+y} or $\sin^2(x)/\sin(x)$ can be simplified as $\sin(x)$. As usual $\log(x) = \ln(x)$ is the **natural log**. Writing $\ln(x)$ instead of $\log(x)$ does not count as a simplification! In the second column, fill in the derivative $f'(x)$ of the expression $f(x)$

	Expression	Simplified expression	Derivative
a)	$\arccos(\cos(x^2))$		
b)	$3^{(4x/\log(3))}$		
c)	$\log(x^3)/(\cos^2(x)+\sin^2(x))$		
d)	$\cos(3x)\tan(3x)$		
e)	$\frac{(2x)^3}{(3x)^4}$		

Solution:

a) simplified as x^2, the derivative is $2x$.

b) simplified as e^{4x} the derivative is $4e^{4x}$.

c) simplified is $3\log(x)$, the derivative is $3/x$.

d) simplified as $\sin(3x)$ the derivative is $3\cos(3x)$.

e) simplified as $(8/81)1/x$, the derivative is $-(8/81)/x^2$.