Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2021

3/5/2021: First hourly

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

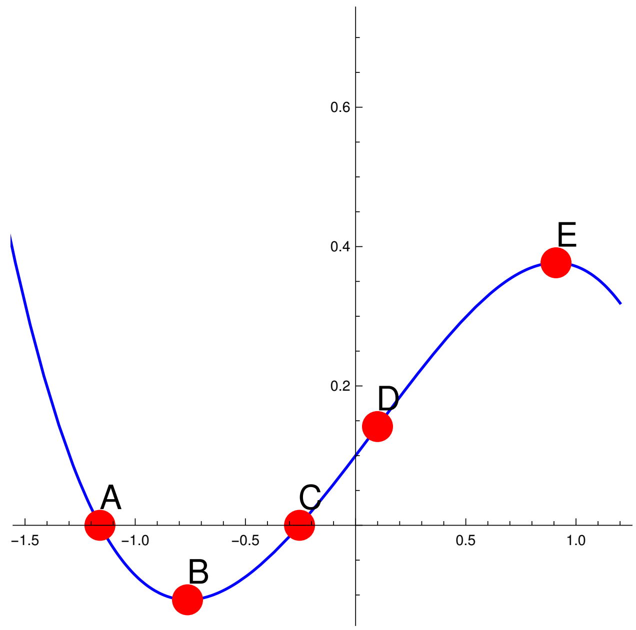
- Solutions are submitted to knill@math.harvard.edu as a PDF, handwritten in one file carrying your name. Capitalize the first letters like in OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False $\cdots$ but you then need to copy the above Honor Code statement and sign.
- No books, calculators, computers, or other electronic aids or online tools or external information are allowed. You can use a double sided page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation. The exam is due at 6 AM on March 6th. Do not communicate with anybody in the class during the exam period.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications are needed.

1)	T	F	The function $f(x) = \log(\exp(x))$ has a root at $x = 0$.
2)	T	F	The "today" function $f(x) = x^3 - x^5 + x^{2021}$ has a real root.
3)	T	F	$\cot(\pi/4) = 1$.
4)	T	F	$f(x) = \|2 + \sin(x)\|$ is differentiable everywhere.
5)	T	F	The chain rule assures that $\frac{d}{dx} \sin(\sin(x)) = \cos(\cos(x)) \cos(x)$.
6)	T	F	$f(x) = 1 + \frac{\sin(x)^2}{x}$ with assumption $f(0) = 1$ is continuous.
7)	T	F	A differentiable function satisfying $f(0) = 1, f(1) = 0$ must have $f'(x)$ be negative somewhere in $[0, 1]$.
8)	T	F	$\sec(x)$ is the inverse function of $\cos(x)$.
9)	T	F	If $f(x)$ is differentiable at 0, and $f(0) = 0$, then $f(f(x))$ is differentiable at 0.
10)	T	F	$f(x) = x^5/(1 + x^5)$ defines an indeterminate form at $x = 0$.
11)	T	F	$f(x) = (\sin(\pi/2+h) - \sin(\pi/2))/h < 0$ for all positive h smaller than $\pi/2$.
12)	T	F	The function $f(x) = x^7$ has a critical point at $x = 0$.
13)	T	F	If f has an inflection point at 0, then $g(x) = 1 - f(x)$ has an inflection point at 0.
14)	T	F	$f(x) = \cot(x^2)$ has a vertical asymptote at $x = 0$.
15)	T	F	$f(x) = \sin(x)$ takes the value $\pi/2$ at some point x.
16)	T	F	If $f(x) = x(x - 1)$, then $Df(x) = f(x + 1) - f(x) = 2x$.
17)	T	F	The function $f(x) = \sin(x) \sin(1/x)$ with the understanding $f(0) = 0$ is continuous everywhere.
18)	T	F	$\frac{d}{dx} \arctan(x^2) = 2x/(1 + x^2)$.
19)	T	F	$\frac{d}{dx} \log(7 - x) = -1/(7 - x)$.
20)	T	F	The derivative of g/f is $(fg' - f'g)/f^2$.

Problem 2) Choice problem (10 points) No justifications are needed.



a) (2 points) List the points A-E which are roots of f.

b) (2 points) List the points A-E that are critical points of f.

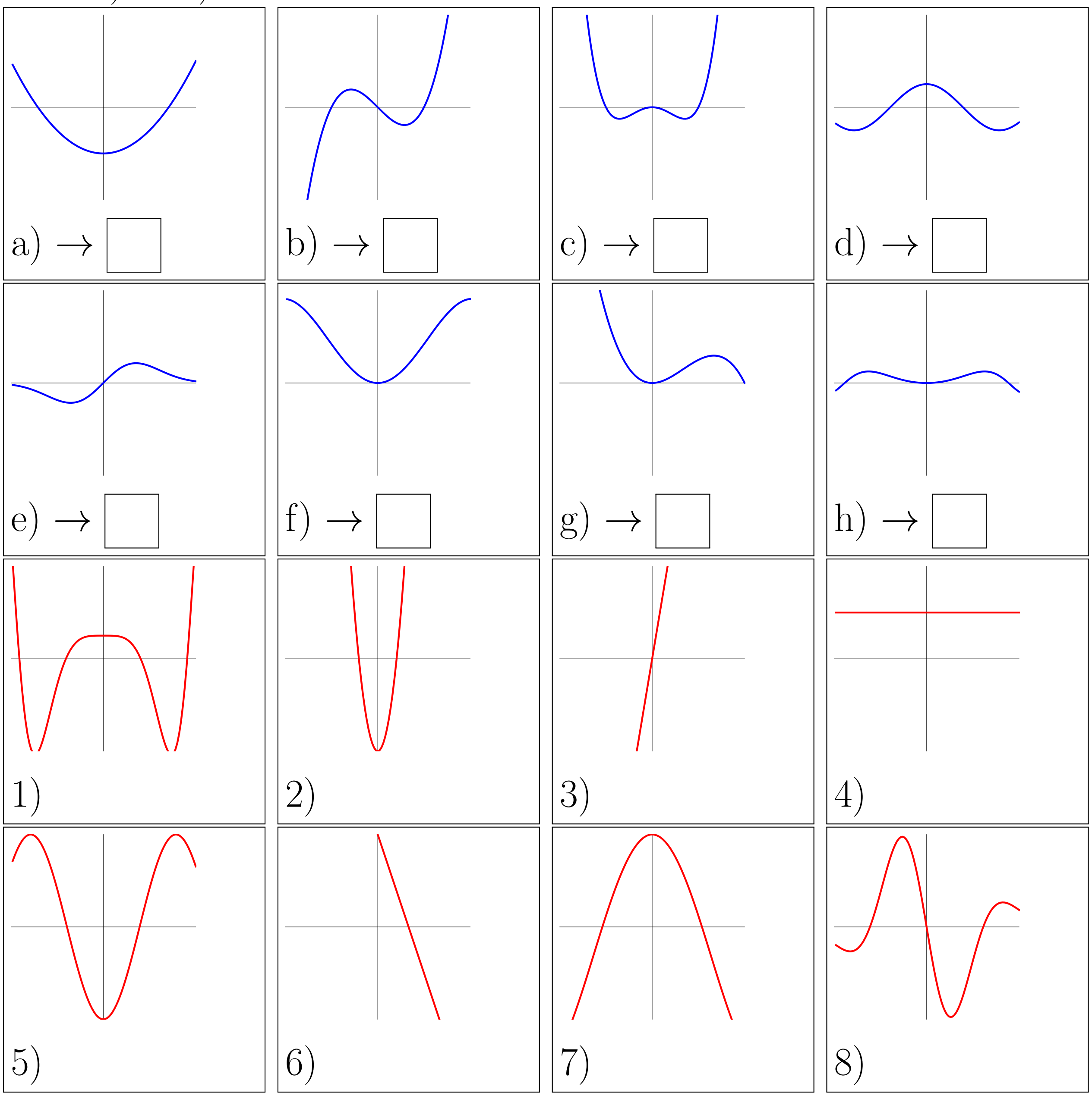
c) (2 points) List the points A-E that are critical points of f'.

d) (2 points) List the critical points A-E that are local maxima.

e) (2 points) List the points A-E of f that are global minima on the given interval.

Problem 3) Matching problem (10 points) No justifications are needed.

Match the functions f a) to h) with the second derivatives functions 1) to 8).



Problem 4) Continuity (10 points)

Which of the following functions are continuous on $[-1, 1]$? As always, we extend continuity to functions for which a continuation is possible to initially not defined points, like $f(x) = (x^2-1)/(x-1)$, which was considered continuous because with $f(1) = 2$, it becomes a continuous function. Make the decision "continuous" or "not continuous" in each of the cases a)-e) and point out any possible x values, which need special attention.

a) (2 points) $f(x) = \cos(\sin(5/x^2))$.

b) (2 points) $f(x) = x^2 \sin(5/x^2)$.

c) (2 points) $f(x) = \frac{\sin^2(x)}{x^2}$

d) (2 points) $f(x) = ||3x| - 4x|$

e) (2 points) $f(x) = 3x/|4x|$

Problem 5) Derivatives (10 points)

Do the required computations for the following functions. In each case, indicate which differentiation rule you use.

a) (2 points) For $f(x) = \sin(x)e^x$, write down $T(x) = x - f(x)/f'(x)$.

b) (2 points) For $f(x) = \log(\sin(x))$, write down $f''(x)$.

c) (2 points) For $f(x) = e^{(e^{\sin(x)})}$, write down $f'(x)$.

d) (2 points) For $f(x) = x^x$, write down $f'(x)$.

e) (2 points) Demonstrate how one can obtain the derivative of $\arcsin(x)$ with the help of the chain rule.

Problem 6) Limits (10 points)

Find the limits $\lim_{x \to 0} f(x)$ for the following functions. As always give details and indicate what method you are using.

a) (2 points) $f(x) = \frac{1-\exp(4x)}{1-\exp(3x)}$.

b) (2 points) $f(x) = \frac{1-x^2}{\cos(3x)}$.

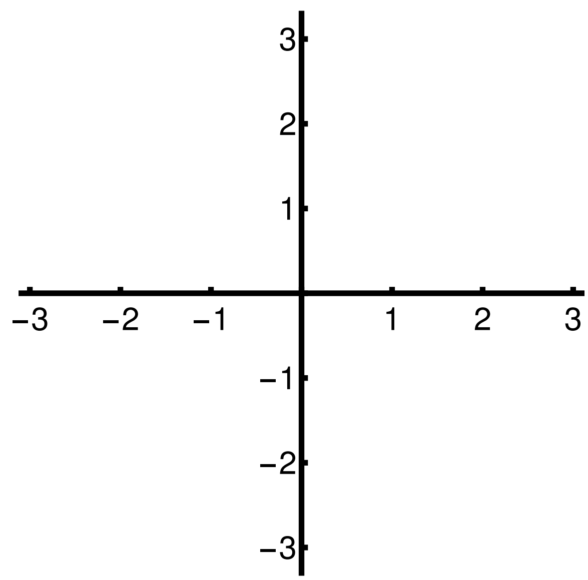
c) (2 points) $f(x) = \frac{3x}{\log(1+4x)}$.

d) (2 points) $f(x) = \frac{\log(x^3)}{\log(x^2)}$.

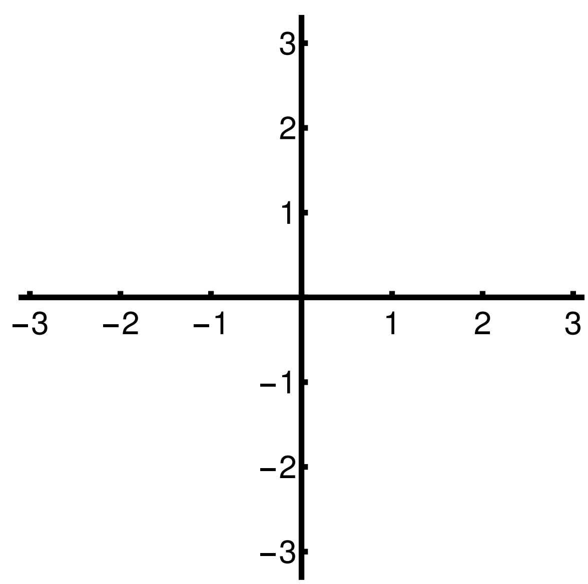
e) (2 points) $f(x) = x(\log(7x))^2$.

Problem 7) Functions (10 points)

a) (5 points) Draw the graph of the function $f(x) = 1 + \sin(3x)$ on $[\pi, \pi]$. Mark the roots, the maxima, minima and inflection points. On paper, please copy the template below exactly!



b) (5 points) Draw the graph of $f(x) = 1/\log|x|$ on $[-3, 3]$. Make sure to indicate what the values at $x = -1, x = 0, x = 1$ are, and draw any possible asymptotes.



Problem 8) Extrema (10 points)

a) (3 points) Find all the critical points of the function

$$f(x) = x^5 - 5x + 7 \ .$$

b) (3 points) Classify the critical points using the second derivative test.

c) (2 points) Find the global maximum and global minimum on $[-2, 2]$.

d) (1 point) Which mathematician proved that a global maximum and a global minimum exists for a continuous function on $[-2, 2]$?

e) (1 point) What is the name of that theorem on global maxima or minima?

Problem 9) Algebra (10 points, 2 points each)

Simplify the following expressions. For example, $3^x 3^y$ can be written fewer letters as 3^{x+y} or $\sin^2(x)/\sin(x)$ can be simplified as $\sin(x)$. As usual $\log(x) = \ln(x)$ is the **natural log**. Writing $\ln(x)$ instead of $\log(x)$ does not count as a simplification! In the second column, fill in the derivative $f'(x)$ of the expression $f(x)$

	Expression	Simplified expression	Derivative
a)	$\arccos(\cos(x^2))$		
b)	$3^{(4x/\log(3))}$		
c)	$\log(x^3)/(\cos^2(x)+\sin^2(x))$		
d)	$\cos(3x)\tan(3x)$		
e)	$\frac{(2x)^3}{(3x)^4}$		