INTRODUCTION TO CALCULUS

MATH 1A

Lecture 33: Calculus and Music

Music is a function

33.1. Calculus matters in music because a piece of music is just a **function**. If you feed a loudspeaker the function $f(t)$ which leads to a displacement of the membrane, the pressure variations in the air are sound waves which then reach your ear, where your ear drum oscillate allowing you to hear the sound. Plotting and playing works the same way. In Mathematica, we can play a function by replacing "Plot" with "Play". For example:

```
Play[ Sin[2Pi 1000 x^2],{x,0,10}]
```

33.2. While the function f contains all the information about the music piece, the computer needs to store this as **data**. A common data format is the ".WAV" file, which contains sampled values of the function, usually with a sample rate of 44100 readings per second. Since our ear does not hear frequencies larger than 20'000 KHz, a sampling rate of 44.1K is good enough by a **theorem of Nyquist-Shannon**. More sophisticated storage possibilities exist. A ".MP3" file for example encodes the function in a compressed way. To get from the sample values $f(k)$ the function back, the **Whittaker-Shannon interpolation formula**

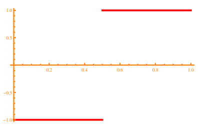
$$f(t) = \sum_{k=1}^{n} f(k)\text{sinc}(t-k)$$

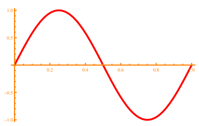
can be used. It involves the **sinc** function $\text{sinc}(x) = \sin(x)/x$ which we have seen earlier.

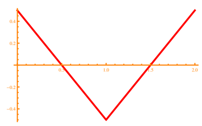
The wave form and hull

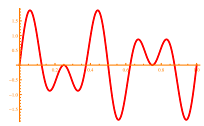
33.3. Periodic signals can serve as **building blocks** of sound. Assume $g(x)$ is a 2π-periodic function, we can generate a sound of 440 Hertz when playing the function $f(x) = g(440 \cdot 2\pi x)$. If the function does not have a smaller period, then we hear the A **tone**. It is a tone with 440 Hertz. We can **modulate** this sound with a **hull function** $h(x)$ and write $f(x) = h(x)g(4402\pi x)$.

Definition: A periodic function g is called a **wave form**.





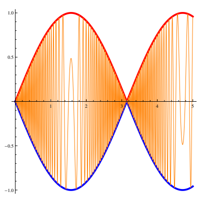




33.4. The wave form makes up the **timbre** of a sound which allows to model music instruments with macroscopic terms like **attack**, **vibrato**, **coloration**, **noise**, **echo**, **reverberation** and other characteristics.

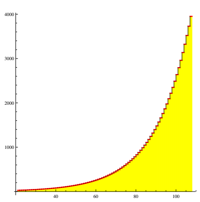
Definition: The **hull function** $h(x)$ is an interpolation of successive local maxima of f.

33.5. For the function $f(x) = \sin(100x)$ for example, the hull function is $h(x) = 1$ For $f(x) = \sin(x)\sin(100x)$ the hull function is $h(x) = |\sin(x)|$. With slight abuse of notation, we sometimes just say $\sin(x)$ is the hull function as the function is sandwiched by the envelopes $\sin(x)$ and $-\sin(x)$.



We can not hear the actual function $f(x)$ because the function changes too fast that we can notice individual vibrations. But we can hear the hull function. We can hear **large scale amplitude** changes like **creshendi** or **diminuendi** or a **vibrato**. When playing two frequencies which are close, one can hear **interference**, the sound analogue of **Moiré patterns** in optics.

The scale



33.6. Western music uses a discrete set of frequencies. This scale is based on the **exponential function**. The frequency f is an exponential function of the scale s. On the other hand, if the frequency is known then the scale number is a logarithm. This is a nice application of the logarithm:

Definition: A frequency f has the **Midi number** $s = 69 + 12 \cdot \log_2(f/440)$. The **piano scale function** or **midi function** gives back $f(s) = 440 \cdot 2^{(s-69)/12}$.

33.7. The Midi tone $s = 100$ for example is a sound of $f = 2637.02$ Hertz (oscillations per second).

The **piano scale function** $f(s) = 440 \cdot 2^{(s-69)/12}$ is an exponential function $f(s) = be^{as}$ which satisfies $f(s+12) = 2f(s)$.

```
midifrequency[m_] := N[440 2^((m - 69)/12)]
```

33.8. A classical piano has 88 keys which scale from 21 to 108. The frequency ranges from $f = 27.5Hz$, the sub-contra-octave A, to the highest $f = 4186.01Hz$, the 5-line octave C.

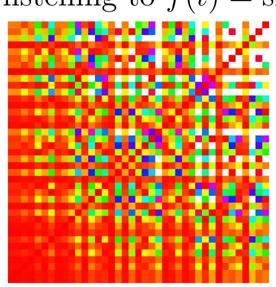
33.9. Filters: a function can be written as a sum of sin and cos functions. Our ear does this so called **Fourier decomposition** automatically. We can so hear melodies, filter out part of the music and hum it.
Pitch and autotune: it is possible to filter out frequencies and adapt their frequency. The popular filter **autotune** moves the frequencies around correcting wrong singing. If 440 Hertz (A) and 523.2 Hertz (C) for example were the only allowed frequencies, the filter would change a function $f(x) = \sin(2\pi 441x) + 4\cos(2\pi 521x)$ to $g(x) = \sin(2\pi 440x) + 4\cos(2\pi 523.2x)$. **Rip and remix**: if f and g are two songs, we can build the average $(f + g)/2$. A composer does this using **tracks**. Different instruments are recorded independently and then mixed together. A guitar $g(t)$, a voice $v(t)$ and a piano $p(t)$ together can form $f(t) = ag(t)+bv(t)+c(p(t)$ with suitably chosen constants a, b, c. **Reverberate and echo**: if f is a song and h is some time interval, we can look at $g(x) = Df(x) = [f(x+h) - f(x)]/h$. For small h, like $h = 1/1000$ the song does not change much because hearing $\sin(kx)$ or $\cos(kx)$ produces the same song. However, for larger h, one can get **reverberate** or **echo** effects.

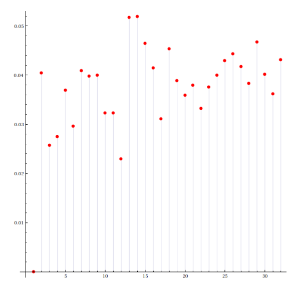
33.10. Mathematics and music have a lot of overlap. Besides wave form analysis and music manipulation operations and symmetry, there are **encoding and compression problems**. A **Diophantine problem** is the question how well a frequency can be approximated by rationals. Why is the **chromatic scale** based on $2^{1/12}$ so effective? **Indian music** for example uses **micro-tones** and a scale of 22. The 12-tone scale has the property that many powers $2^{k/12}$ are close to rational numbers. This can be quantified with the **scale fitness**

$$M(n) = \sum_{k=1}^{n} \mathrm{min}_{\mathrm{p,q}} |2^{k/n} - \frac{p}{q}| G(p/q)$$

where $G(n/m)$ is Euler's **gradus suavitatis** (="degree of sweetness") defined as $G(n/m) = 1 + \sum_{p|n*m}(p-1)$ in which the sum runs over all prime factors p of $n * m$. For example $G(3/4) = 1 + (2-1) + (2-1) + (3-1)$ because $3 * 4 = 12 = 2 * 2 * 3$.

33.11. The figure below illustrates why the 12-tone scale minimizes $M(n)$. We could also replace the concept of octave. Stockhausen experimented with replacing 2 with 5 and used the **Stockhausen scale** $5^{k/25}$. It is $f(t) = \sin(2\pi t 440 \cdot 5^{[t]/25})$, where $[t]$ is the largest integer smaller than t. The familiar **12-tone scale** can be admired by listening to $f(t) = \sin(2\pi t 440 \cdot 2^{[t]/12})$.





Example: The perfect fifth 3/2 has the gradus suavitatis $1 + E(6) = 1 + 2 = 3$ which is the same than the perfect fourth 4/3 for which $1+E(12) = 1+(2-1)(3-1)$. You can listen to the perfect fifth $f(x) = \sin(1000x)+\sin(1500x)$ or the perfect fourth $\sin(1000x)+\sin(1333x)$ and here is a function representing an **accord** with four notes $\sin(1000x) + \sin(1333x) + \sin(1500x) + \sin(2000x)$.

Homework

Problem 33.1: Modulation. Draw and play the following function

$$f(x) = \cos(4000x) - \cos(4011x)$$

for three seconds. You can your AI and just tell it in words what you want to do! How many up and downs to you hear in the hull? Do the same for

$$f(x) = \cos(4000x) - \cos(4021x) \ .$$

Here is how to play a function with Mathematica or Wolfram alpha:

```
Play[Cos[x] Sin[Exp[2 x]]/x, {x, 0,9}]
play  sin(1000 x)
```

Problem 33.2: Amplitude modulation (AM): If you listen to $f(x) = |\cos(x^2)|\sin(1000x)$ you hear an amplitude change. Draw the hull function or listen to it and count how many increases in amplitudes to you hear in 10 seconds.

Problem 33.3: Other tonal scales, Midi number: As a creative musician, we create our own tonal scale. You decide to take the 8'th root of 3 as your basic frequency change from one tone to the next.
a) After how many tonal steps has the frequency f tripled?
b) Build the midi function and then write down the inverse for your tonal scale.

Problem 33.4: a) What is the frequency of the Midi number $s = 22$?
b) Which midi number belongs to the frequency $f = 2060Herz$?

Problem 33.5: Gradus Suavitatis. a) What is the gradus suavitatis of 49/64? b) What is the gradus suavitatis of 541/221?

Oliver Knill, knill@math.harvard.edu, Math 1a, Harvard College, Spring 2021