INTRODUCTION TO CALCULUS

MATH 1A

Unit 29: Trig Substitution

Lecture

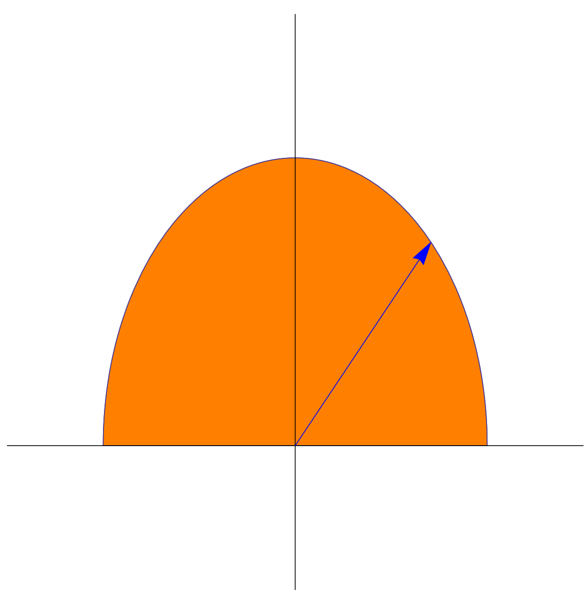
29.1. A **trig substitution** is a substitution, where x is a trigonometric function of u or u is a trigonometric function of x. Here is an important example:

Example: The area of a half circle of radius 1 is given by the integral

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx \ .$$

Solution. Write $x = \sin(u)$ so that $\cos(u) = \sqrt{1-x^2}$. $dx = \cos(u)du$. We have $\sin(-\pi/2) = -1$ and $\sin(\pi/2) = 1$ the answer is

$$\int_{-\pi/2}^{\pi/2} \cos(u)\cos(u)du = \int_{-\pi/2}^{\pi/2} (1+\cos(2u))/2 = \frac{\pi}{2} \ .$$



29.2. Let us do the same computation for a general radius r:

Example: Compute the area of a half disc of radius r which is given by the integral

$$\int_{-r}^{r} \sqrt{r^2-x^2}\, dx \ .$$

Solution. Write $x = r\sin(u)$ so that $r\cos(u) = \sqrt{r^2 - x^2}$ and $dx = r\cos(u)\ du$ and $r\sin(-\pi/2) = -r$ and and $r\sin(\pi/2) = r$. The answer is

$$\int_{-\pi/2}^{\pi/2} r^2\cos^2(u)\ du = r^2\pi/2\ .$$

29.3. Here is an example, we know already how to integrate. But now we derive it from scratch:
Example: Find the integral

$$\int \frac{dx}{\sqrt{1-x^2}}\ .$$

We know the answer is $\arcsin(x)$. How can we do that without knowing? **Solution.** We can do it also with a trig substitution. Try $x = \sin(u)$ to get $dx = \cos(u)\ du$ and so

$$\int \frac{\cos(u)\ du}{\cos(u)} = u = \arcsin(x) + C\ .$$

29.4. In the next example, $x = \tan(u)$ works. You have to be told that first as it is hard to come up with the idea:
Example: Find the following integral:

$$\int \frac{dx}{x^2\sqrt{1+x^2}}$$

by using the substitution $x = \tan(u)$. **Solution.** Then $1 + x^2 = 1/\cos^2(u)$ and $dx = du/\cos^2(u)$. We get

$$\int \frac{du}{\cos^2(u)\tan^2(u)(1/\cos(u))} = \int \frac{\cos(u)}{\sin^2(u)}\ du = -1/\sin(u) = -1/\sin(\arctan(x))\ .$$

29.5. For trig substitution, the following basic trig identity is important:

$\cos^2(u) + \sin^2(u) = 1$

Depending on whether dividing by $\sin^2(u)$ or $\cos^2(u)$, we get

$1 + \tan^2(u) = 1/\cos^2(u)$, $1 + \cot^2(u) = 1/\sin^2(u)$

These identities come handy: lets look at more examples:
Example: Evaluate the following integral

$$\int x^2/\sqrt{1-x^2}\ dx\ .$$

Solution: Substitute $x = \cos(u), dx = -\sin(u)\ du$ and get

$$\int -\frac{\cos^2(u)}{\sin(u)}\ \sin(u)du = -\int \cos^2(u)\ du = -\frac{u}{2} - \frac{\sin(2u)}{4} + C = -\frac{\arcsin(x)}{2} + \frac{\sin(2\arcsin(x))}{4} + C\ .$$

Example: Evaluate the integral

$$\int \frac{dx}{(1+x^2)^2} \; .$$

Solution: we make the substitution $x = \tan(u), dx = du/(\cos^2(u))$. Since $1 + x^2 = \cos^{-2}(u)$ we have

$$\int \frac{dx}{(1+x^2)^2} = \int \cos^2(u)\; du = (u/2) + \frac{\sin(2u)}{4} + C = \frac{\arctan(u)}{2} + \frac{\sin(2\arctan(u))}{4} + C \; .$$

29.6. Here is an other prototype problem:

Example: Find the anti derivative of $1/\sin(x)$. **Solution**: We use the substitution $u = \tan(x/2)$ which gives $x = 2\arctan(u), dx = 2du/(1+u^2)$. Because $1 + u^2 = 1/\cos^2(x/2)$ we have

$$\frac{2u}{1+u^2} = 2\tan(x/2)\cos^2(x/2) = 2\sin(x/2)\cos(x/2) = \sin(x) \; .$$

Plug this into the integral

$$\int \frac{1}{\sin(x)}\; dx = \int \frac{1+u^2}{2u} \frac{2du}{1+u^2} = \int \frac{1}{u}\; du = \log(u) + C = \log(\tan(\frac{x}{2})) + C \; .$$

Unlike before, where x is a trig function of u, now u is a trig function of x. This example shows that the substitution $u = \tan(x/2)$ is magic. It leads to the following formulas. We can call it the **magic box**:

$u = \tan(x/2)$
$\boxed{1}$ $dx = \frac{2du}{(1+u^2)}$
$\boxed{2}$ $\sin(x) = \frac{2u}{1+u^2}$
$\boxed{3}$ $\cos(x) = \frac{1-u^2}{1+u^2}$

29.7. The magic box allows us to reduce any rational function involving trig functions to rational functions. We can also let the machine do it: [1]

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u=Tan[x/2]; Simplify[D[u,x]==(1+u^2)/2]
Simplify[Cos[x]== (1-u^2)/(1+u^2)]
Simplify[Sin[x]== 2u/(1+u^2)]
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Any function $p(x)/q(x)$ where p, q are trigonometric polynomials can now be integrated using elementary functions.

[1]As human: $\boxed{1}$ differentiate to get $du = dx/(2\cos^2(x/2)) = dx(1+u^2)/2$. $\boxed{2}$ use double angle $\sin(x) = 2\tan(x/2)\cos^2(x/2)$ and then $1/\cos^2(x/2) = 1 + \tan^2(x/2)$. $\boxed{3}$ use double angle $\cos(x) = \cos^2(x/2) - \sin^2(x/2) = (1 - \sin^2(x/2)/\cos^2(x/2))\cos^2(x/2)$ and again $1/\cos^2(x/2) = 1 + \tan^2(x/2)$.

29.8. It is usually a lot of work, but here is an example:
Example: To find the integral

$$\int \frac{\cos(x) + \tan(x)}{\sin(x) + \cot(x)}\, dx$$

for example, we replace $dx, \sin(x), \cos(x), \tan(x) = \sin(x)/\cos(x), \cot(x) = \cos(x)/\sin(x)$ with the above formulas we get a rational expression which involves u only This gives us an integral $\int p(u)/q(u)\, du$ with polynomials p, q. In our case, this would simplify to

$$\int \frac{2u\left(u^4 + 2u^3 - 2u^2 + 2u + 1\right)}{(u-1)(u+1)\left(u^2+1\right)\left(u^4 - 4u^2 - 1\right)} du$$

The method of partial fractions provides us then with the solution.

Homework

Problem 28.1: Find the anti-derivative:

$$\int \sqrt{1 - 16x^2}\, dx\ .$$

Problem 28.2: Find the anti-derivative:

$$\int (1 - x^2)^{3/2}\, dx\ .$$

Problem 28.3: Find the anti-derivative:

$$\int \frac{\sqrt{1 - x^2}}{x^2}\, dx\ .$$

Problem 28.4: Integrate

$$\int \frac{dx}{1 + \sin(x)}\ .$$

Use the substitution $u = \tan(x/2)$ and use the magic box.

Problem 28.5: Use the magic.
a) Compute

$$\int \frac{\tan(x/2) dx}{\cos(x)}\ .$$

b) Do it in the same way:

$$\int \frac{\tan^2(x/2) dx}{\cos(x)}\ .$$

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