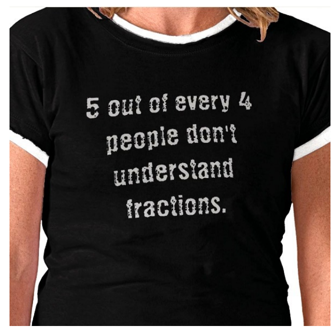
INTRODUCTION TO CALCULUS

MATH 1A

Unit 26: Partial fractions

Lecture



The method of partial fractions is not really about integration. It is about algebra. We have learned how to integrate polynomials like $x^4 + 5x + 3$. What about rational functions? We will see here that they are a piece of cake if you know a bit about algebra.

26.1. Lets see what we know already:

- We also know that integrating $1/x$ gives $\log(x)$. We can for example integrate
$$\int \frac{1}{x-6}\, dx = \log(x-6) + C \; .$$
- We also have learned how to integrate $1/(1+x^2)$. It was an important integral:
$$\int \frac{1}{1+x^2}\, dx = \arctan(x) + C \; .$$
Using substitution, we can do more like
$$\int \frac{dx}{1+4x^2} = \int \frac{du/2}{1+u^2} = \arctan(u)/2 = \arctan(2x)/2 \; .$$
- We also know how to integrate functions of the type $x/(x^2+c)$ using substitution. We can write $u = x^2 + c$ and get $du = 2x dx$ so that
$$\int \frac{x}{x^2+c}\, dx = \int \frac{1}{2u}\, du = \frac{\log(x^2+c)}{2} \; .$$
- Also functions $1/(x+c)^2$ can be integrated using substitution. With $x + c = u$ we get $du = dx$ and
$$\int \frac{1}{(x+c)^2}\, dx = \int \frac{1}{u^2} du = -\frac{1}{u} + C = -\frac{1}{x+c} + C \; .$$

26.2. We would love to be able to integrate any rational function

$$f(x) = \frac{p(x)}{q(x)} ,$$

where p, q are polynomials. This is where **partial fractions come in**. The idea is to write a rational function as a sum of fractions we know how to integrate. The above examples have shown that we can integrate $a/(x+c), (ax+b)/(x^2+c), a/(x+c)^2$ and cases, which after substitution are of this type.

Definition: The **partial fraction method** writes $p(x)/q(x)$ as a sum of functions of the above type which we can integrate.

26.3. This is an algebra problem. Here is an important special case:

In order to integrate $\int \frac{1}{(x-a)(x-b)} \, dx$, write

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} .$$

and solve for A, B.

26.4. In order to solve for A, B, write the right hand side as one fraction again

$$\frac{1}{(x-a)(x-b)} = \frac{A(x-b) + B(x-a)}{(x-a)(x-b)} .$$

We only need to look at the nominator:

$$1 = Ax - Ab + Bx - Ba .$$

In order that this is true we must have $A + B = 0, Ab - Ba = 1$. This allows us to solve for A, B.

Examples

Example: To integrate $\int \frac{2}{1-x^2} \, dx$ we can write

$$\frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$$

and integrate each term

$$\int \frac{2}{1-x^2} = \log(1+x) - \log(1-x) .$$

Example: Integrate $\frac{5-2x}{x^2-5x+6}$. **Solution.** The denominator is factored as $(x-2)(x-3)$. Write

$$\frac{5-2x}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2} .$$

Now multiply out and solve for A, B:

$$A(x-2) + B(x-3) = 5 - 2x .$$

This gives the equations $A+B=-2, -2A-3B=5$. From the first equation we get $A=-B-2$ and from the second equation we get $2B+4-3B=5$ so that $B=-1$ and so $A=-1$. We have not obtained

$$\frac{5-2x}{x^2-5x+6} = -\frac{1}{x-3} - \frac{1}{x-2}$$

and can integrate:

$$\int \frac{5-2x}{x^2-5x+6}\, dx = -\log(x-3) - \log(x-2) \; .$$

Actually, we could have got this one also with substitution. How?

Example: Integrate $f(x) = \int \frac{1}{1-4x^2}\, dx$. **Solution.** The denominator is factored as $(1-2x)(1+2x)$. Write

$$\frac{A}{1-2x} + \frac{B}{1+2x} = \frac{1}{1-4x^2} \; .$$

We get $A = 1/4$ and $B = -1/4$ and get the integral

$$\int f(x)\, dx = \frac{1}{4}\log(1-2x) - \frac{1}{4}\log(1+2x) + C \; .$$

26.5. There is a fast method to get the coefficients:

If a is different from b, then the coefficients A, B in

$$\frac{p(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} \; ,$$

are

$$A = \lim_{x\to a}(x-a)f(x) = p(a)/(a-b), \; B = \lim_{x\to b}(x-b)f(x) = p(b)/(b-a) \; .$$

Proof. If we multiply the identity with $x-a$ we get

$$\frac{p(x)}{(x-b)} = A + \frac{B(x-a)}{x-b} \; .$$

Now we can take the limit $x \to a$ without peril and end up with $A = p(a)/(x-b)$.

26.6. Cool, isn't it? This **Hospital method** or **residue method** saves time especially with many functions where we would a complicated system of linear equations would have to be solved. I highly recommend you use this method.

Math is all about elegance. Avoid complicated methods if simple ones are available.

26.7. Here are examples:

Example: Find the anti-derivative of $f(x) = \frac{2x+3}{(x-4)(x+8)}$. **Solution.** We write

$$\frac{2x+3}{(x-4)(x+8)} = \frac{A}{x-4} + \frac{B}{x+8}$$

Now $A = \frac{2*4+3}{4+8} = 11/12$, and $B = \frac{2*(-8)+3}{(-8-4)} = 13/12$. We have

$$\frac{2x+3}{(x-4)(x+8)} = \frac{(11/12)}{x-4} + \frac{(13/12)}{x+8} \ .$$

The integral is

$$\frac{11}{12}\log(x-4) + \frac{13}{12}\log(x+8) \ .$$

Example: Find the anti-derivative of $f(x) = \frac{x^2+x+1}{(x-1)(x-2)(x-3)}$. **Solution.** We write

$$\frac{x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Now $A = \frac{1^2+1+1}{(1-2)(1-3)} = 3/2$ and $B = \frac{2^2+2+1}{(2-1)(2-3)} = -7$ and $C = \frac{3^2+3+1}{(3-1)(3-2)} = 13/2$. The integral is

$$\frac{3}{2}\log(x-1) - 7\log(x-2) + \frac{13}{2}\log(x-3) \ .$$

Homework

As we had a wellness day break, this homework mixes problems from unit 25 and 26.

Problem 26.1: Do without the tabular method:
a) $\int xe^{7x+1}\, dx$.
b) $\int x^2 \log(x)\, dx$.

Problem 26.2: $\int (x-1)^7 \sin(3x)\, dx$. Do this with the tabular method.

Problem 26.3: a) $\int \cos(3x)\sin(5x)\, dx$ (Merry go round).
b) $\int \frac{1}{x^2-14x+45}\, dx$
c) $\int \frac{2}{x^2-9}\, dx$

Problem 26.4: $\int \frac{x^3-x+1}{x^2-1}\, dx$. Subtract first a polynomial.

Problem 26.5: $\int \frac{1}{(x+1)(x-1)(x+7)(x-3)}\, dx$.