Unit 24: Substitution

Lecture

24.1. We know so far how to integrate functions like e^{6x} or $1/(1+x)$. The technique of **substitution** allows us to find more complicated anti-derivatives. If we differentiate the function $\sin(x^2)$ and use the chain rule, we get $\cos(x^2)2x$. The fundamental theorem of calculus tells us therefore that the anti-derivative of $\cos(x^2)2x$ is

$$\int \cos(x^2)2x \, dx = \sin(x^2) + C \; .$$

24.2. How can we see the integral without knowing the result already? Here is a very important case:

If $f(x) = g(u(x))u'(x)$, then the anti-derivative of f is $G(u(x)) + C$, where G is the anti-derivative of g.

Example: Find the anti derivative of

$$f(x) = e^{x^4+x^2}(4x^3 + 2x) \; .$$

Solution: It is $e^{x^4+x^2} + C$.

Example: Find

$$\int \sqrt{x^5+1}x^4 \, dx \; .$$

Solution. Try $(x^5+1)^{3/2}$ and differentiate. This gives $15/2$ of what we have. Therefore $F(x) = (2/15)(x^5+1)^{3/2}$.

Example: Find the anti-derivative of

$$\frac{\log(x)}{x} \; .$$

Solution: We spot that $1/x$ is the derivative of $\log(x)$. The anti-derivative is $\log(x)^2/2 + C$.

24.3. Writing down the function and adjusting a constant is the **"speedy rule"**:

If $\int f(ax+b)\,dx = F(ax+b)/a$ where F is the anti derivative of f.

Example: $\int \sqrt{x+1}\,dx$. **Solution:** $(x+1)^{3/2}(2/3)+C$.
Example: $\int \frac{1}{1+(5x+2)^2}\,dx$. **Solution:** $\arctan(5x+2)(1/5)+C$.

24.4. The method of substitution formalizes this: A) select part of the formula, call it u. B) then write $du = u'dx$. C) replace dx with du/u'. D) If all terms x have disappeared, integrate. E) Back substitute the variable x. If things should not work, go back to A) and try an other u.

$$\int f(\, \mathrm{u(x)}\,)\, \mathrm{u'(x)} \quad \mathrm{dx} = \int g(\, \mathrm{u}\,)\, \mathrm{du}\ .$$

24.5. We aim to end up with an integral $\int g(u)\,du$ which does not involve x anymore. Finally, after integration of this integral, do a **back-substitution**: replace the variable u again with the function $u(x)$.
Example: Find the anti-derivative of $\int \log(x)/x\,dx$. **Solution:** Pick $u = log(x), du = (1/x)dx$. Because $dx = xdu$, we get $\int udu = u^2/2 + C$. Back substitute to get $\log^2(x)/2 + C$.
Example: Find the anti-derivative

$$\int \log(\log(x))\frac{1}{\log(x)x}\,dx\ .$$

Solution: Try $u = \log(x)$ and $du = (1/x)dx$, then plug this into the formula. It gives $\int \log(u)/u\,du$. We have just solved this integral before and got $\log(u)^2/2 + C$.
Example: Solve the integral

$$\int \frac{x}{1+x^4}\,dx\ .$$

Solution: Substitute $u = x^2, du = 2xdx$ gives $(1/2)\int du/(1+u^2)\,du = (1/2)\arctan(u) = (1/2)\arctan(x^2) + C$.

Example: What is the anti-derivative of $\sin(\sqrt{x})/\sqrt{x}$?
Solution. Try $u = \sqrt{x}, x = u^2, dx = 2udu$. The result is $-2\cos(\sqrt{(}x)) + C$.

24.6. Here is an example that is more challenging
Example: Solve the integral

$$\int \frac{x^3}{\sqrt{x^2+1}} \, dx \; .$$

Solution. Trying $u = \sqrt{x^2+1}$ does not work. Try $u = x^2 + 1$, then $du = 2xdx$ and $dx = du/(2\sqrt{u-1})$. Substitute this in to get

$$\int \frac{\sqrt{u-1}^3}{2\sqrt{u-1}\sqrt{u}} \, du = \int \frac{(u-1)}{2\sqrt{u}} = \int u^{1/2}/2 - u^{-1/2}/2 \; du = u^{3/2}/3 - u^{1/2} = \frac{(x^2+1)^{3/2}}{3} - (x^2+1)^{1/2} \; .$$

24.7. When doing **definite integrals** $\int_a^b f(x) \, dx$, we could find the anti-derivative as described and then fill in the boundary points. Substituting the boundaries directly accelerates the process since we do not have to substitute back to the original variables:

$$\int_a^b g(u(x))u'(x) \, dx = \int_{u(a)}^{u(b)} g(u) \, du \; .$$

Proof. This identity follows from the fact that the right hand side is $G(u(b)) - G(u(a))$ by the fundamental theorem of calculus. The integrand on the left has the anti derivative $G(u(x))$. Again by the fundamental theorem of calculus the integral leads to $G(u(b)) - G(u(a))$.

Example: Find the anti-derivative of $\int_0^2 \sin(x^3 - 1)x^2 \, dx$. **Solution**:

$$\int_{x=0}^{x=2} \sin(x^3+1)x^2 \, dx \; .$$

Solution: Use $u = x^3 + 1$ and get $du = 3x^2dx$. We get

$$\int_{u=1}^{u=9} \sin(u)du/3 = (1/3)\cos(u)|_1^9 = [-\cos(9) + \cos(1)]/3 \; .$$

Example: $\int_0^1 \frac{1}{5x+1} \, dx = [\log(u)]/5|_1^6 = \log(6)/5$.
Example: $\int_3^5 \exp(4x - 10) \, dx = [\exp(10) - \exp(2)]/4$.

24.8. Substituting the bounds can sometimes be a bit tricky. An alternative way is to find first the anti derivative and then plug in the original bounds. Avoiding substituting the bounds actually is often the preferred way.
Example: $\int_3^5 \exp(4x - 10) \, dx = F(5) - f(3)$, where $F(x) = \exp(4x - 10)/4$.

Homework

Problem 24.1: Find the following anti-derivatives.
a) $\int x^2 \sin(x^3)\, dx$
b) $\int e^{x^6+x}(6x^5+1)\, dx$
c) $-\cos(\sin(3x))/3$
d) $e^{\tan(2x)}/\cos^2(2x)$.

Problem 24.2: Compute the following definite integrals. It is fine to find first the anti-derivative and only at the end place the bounds:
a) $\int_1^2 \sqrt{x^5+x}(5x^4+1)\, dx$
b) $\int_0^{\sqrt{\pi}} 6\sin(x^2)x\, dx$.
c) $\int_e^{e^2} \frac{\sqrt{\log(x)}}{x}\, dx$.
d) $\int_0^1 \frac{5x}{\sqrt{1+x^2}}\, dx$.

Problem 24.3: Find the definite integral

$$\int_e^{2e} \frac{dx}{\sqrt{\log(x)}x} .$$

Problem 24.4: a) Find the indefinite integral

$$\int \frac{x^5}{\sqrt{x^2+1}}\, dx .$$

b) Find the anti-derivative of

$$f(x) = \frac{1}{x(1+\log(x)^2)} .$$

Problem 24.5: a) Find the anti-derivative of $\cos(x^3)/e^{\sin(x^3)}x^2$.
b) Find the anti-derivative of $\cot(\sqrt{x})/\sqrt{x}$.

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