**INTRODUCTION TO CALCULUS**

MATH 1A

# Unit 17: Riemann Integral

## Lecture

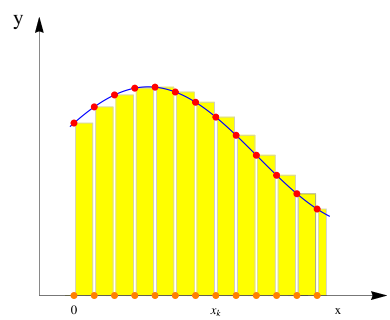
**17.1.** In this lecture, we define the definite integral $\int_0^x f(t)\, dt$ if $f$ is a differentiable function. We then compute it for some basic functions. We have previously defined the **Riemann sums**

$$Sf(x) = h[\, f(0) + f(h) + f(2h) + \cdots + f(kh)]\ ,$$

where $k$ is the largest integer such that $kh < x$. Lets write $S_n$ if we want to stress that the parameter $h = 1/n$ was used in the sum. We define the **Riemann integral** as the limit of these sums $S_n f$, when the **mesh size** $h = 1/n$ goes to zero.

**Definition:** Define

$$\int_0^x f(t)\, dt = \lim_{n \to 0} S_n f(x)\ .$$




**17.2.** A very important result is that

For any continuous function, the limit exists.

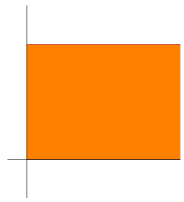
It is easier to see when $f$ is $f$ is differentiable as one can then estimate the error. There are $n$ little pieces which are each of area $\leq M/n$, where $M$ is the maximal slope that $f$ can have in the given interval.

For non-negative $f$, the value $\int_0^x f(x)\, dx$ is the **area between the x-axis and the graph** of $f$. For general $f$, it is a **signed area**, the difference between two areas.

**17.3.** The Riemann integral is the limit $h \sum_{x_k = kh \in [0,x)} f(x_k)$. It converges to the area under the curve for all **continuous** functions. In probability theory, one uses also an other integral, the **Lebesgue integral**. It can be defined as the limit $\frac{1}{n}\sum_{k=1}^{n} f(x_k)$ where $x_k$ are **random points** in $[0,x]$. This is a **Monte-Carlo integral** definition of the Lebesgue integral.

**17.4.** Riemann also looked also at points $x_0 < x_1 < \cdots < x_n$ $[0,x]$ such that the maximal distance $(x_{k+1} - x_k)$ between neighboring $x_j$ goes to zero. The Riemann sum is then $S_n f = \sum_k f(y_k)(x_{k+1} - x_k)$, where $y_k$ is arbitrarily chosen inside the interval $(x_k, x_{k+1})$. For continuous functions, the limiting result is the same the $Sf(x)$ sum done here. There are numerical reasons to allow more general partitions because it allows to adapt the mesh size: use more points where the function is complicated.

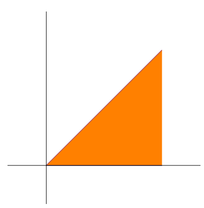
**Example:** If $f(x) = c$ is constant, then $\int_0^x f(t)\,dt = cx$. We can see also that $cnx/n \le S_n f(x) \le c(n+1)x/n$.



**Example:** Let $f(x) = cx$. The area is half of a rectangle of width $x$ and height $cx$ so that the area is $cx^2/2$. Adding up the Riemann sum is more difficult. Let $k$ be the largest integer smaller than $xn = x/h$. Then

$$S_n f(x) = \frac{1}{n}\sum_{j=1}^{k} \frac{cj}{n} = \frac{ck(k+1)/2}{n^2} .$$

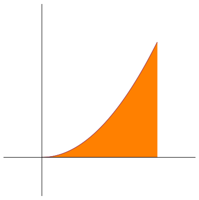
Taking the limit $n \to \infty$ and using that $k/n \to x$ shows that $\int_0^x f(t)\,dt = cx^2/2$.

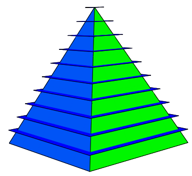


**Example:** Let $f(x) = x^2$. In this case,we can not see the numerical value of the area geometrically. But since we have computed $S[x^2]$ in the first lecture of this course and seen that it is $[x^3]/3$ and since we have defined $S_h f(x) \to \int_0^x f(t)\,dt$ for $h \to 0$ and $[x^k] \to x^k$ for $h \to 0$, we know that

$$\int_0^x t^2\,dt = \frac{x^3}{3} .$$

This example actually computes the **volume of a pyramid** which has at distance $t$ from the top an area $t^2$ cross section. Think about $t^2 dt$ as a slice of the pyramid of area $t^2$ and height $dt$. Adding up the volumes of all these slices gives the volume.

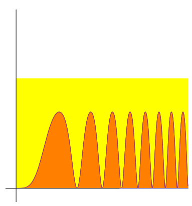




**Linearity of the integral** (see homework) $\int_0^x f(t)+g(t)\ dt = \int_0^t f(t)\ dt + \int_0^x g(t)\ dt$ and $\int_0^x \lambda f(t)\ dt = \lambda \int_0^x f(t)\ dt$.

**Upper bound**: If $0 \leq f(x) \leq M$ for all $x$, then $\int_0^x f(t)\ dt \leq Mx$.

**Example:** $\int_0^x \sin^2(\sin(\sin(t)))/x\ dt \leq x$. **Solution**. The function $f(t)$ inside the interval is non-negative and smaller or equal to 1 The graph of $f$ is therefore contained in a rectangle of width $x$ and height 1.



**17.5.** We see that if two functions are close then their difference is a function which is included in a small rectangle and therefore has a small integral:

If $f$ and $g$ satisfy $|f(x) - g(x)| \leq c$, then
$$\int_0^x |f(x) - g(x)|\ dx \leq cx\ .$$

**17.6.** We know identities like $S_n[x]_h^n = \frac{[x]_h^{n+1}}{n+1}$ and $S_n \exp_h(x) = \exp_h(x)$ already. Since $[x]_h^k - [x]^k \to 0$ we have $S_n[x]_h^k - S_n[x]^k \to 0$ and from $S_n[x]_h^k = [x]_h^{k+1}/(k+1)$. The other equalities are the same since $\exp_h(x) = \exp(x) \to 0$. This gives us:

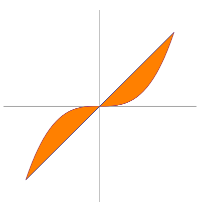
| | |
|---|---|
| $\int_0^x t^n\ dt = \frac{x^{n+1}}{n+1}$ | $\int_0^x \cos(t)\ dt = \sin(x)$ |
| $\int_0^x e^t\ dt = e^x - 1$ | $\int_0^x \sin(t)\ dt = 1 - \cos(x)$ |

## Homework

In the following homework you can use that $\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$ if $F$ is a function which satisfies $F'(x) = f(x)$. We have already verified the identity for sums.

**Problem 17.1:** a) What is the integral $\int_0^1 4x^{24}\,dx$?
b) Find the integral $\int_0^{\log(2)} 5e^t\,dt$.
c) Calculate $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}}\,dx$.
d) Find $\int_0^{\pi/2} \cos^2(t)\,dt$.
e) Find $\int_0^{\pi/2} \sin^4(t)\,dt$.

**Problem 17.2:** The region enclosed by the graph of $x$ and the graph of $x^5$ has a propeller type shape. Find its (positive) area.



**Problem 17.3:** Make a geometric picture for each of the following statements (which are rules for integration):

- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$.
- $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$.
- $\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$.

**Problem 17.4:** Here are some more challenging integrals. Maybe you have to guess or remember some old computations, maybe even look at the exam. a) $\int_0^2 x^x(1+\log(x))\,dx$
b) $\int_0^1 (3/2)\sqrt{1+x}\,dx$
c) $\int_0^{\sqrt{\log(2)}} 4xe^{-x^2}\,dx$
d) $\int_1^e 5\log(x)/x\,dx$

**Problem 17.5:** In this problem, it is crucial that you plot the function first. Split the integral up into parts. Find $\int_{-1}^4 f(x)\,dx$ for $f(x) = |x - |x-2||$.