INTRODUCTION TO CALCULUS

MATH 1A

Unit 15: Review

Major points

f is **continuous** at a if there is $b = f(a)$ such that $\lim_{x \to a} f(x) = b$ for every a. The intermediate value theorem: $f(a) > 0, f(b) < 0$ implies f having a root in (a, b).

$f'(x) = 0, f''(x) > 0$ then x is **local min**. $f'(x) = 0,'' (x) < 0$ then x is **local max**. For **global minima or maxima**, compare local extrema and boundary values.

If f changes sign we have a **root** $f = 0$, if f' changes sign, we have a **critical point** $f' = 0$ if f'' changes sign, we have an **inflection points**. A function is **even** if $f(-x) = f(x)$, and **odd** if $f(-x) = -f(x)$.

If $f' > 0$ then f is increasing, if $f' < 0$ it is decreasing. If $f''(x) > 0$ it is **concave up**, if $f''(x) < 0$ it is **concave down**. If $f'(x) = 0$ then f has a horizontal tangent.

Hosptal's theorem applies for indeterminate forms $0/0$ or ∞/∞. In that case, $\lim_{x \to a} f(x)/g(x)$, where $f(a) = g(a) = 0$ or $f(a) = g(a) = \infty$ with $g'(a) \neq 0$ are given by $f'(a)/g'(a)$.

With $Df(x) = (f(x+h) - f(x))/h$ and $S(x) = h(f(0) + f(2h) + \cdots + f((k-1)h))$ we have a **preliminary fundamental theorem of calculus** $SDf(kh) = f(kh) - f(0)$ and $DS(f(kh)) = f(kh)$.

Roots of $f(x)$ with $f(a) < 0, f(b) > 0$ can be obtained by the dissection method by applying the **Newton map** $T(x) = x - f(x)/f'(x)$ again and again.

Algebra reminders

Healing:	$(a + b)(a - b) = a^2 - b^2$ or $1 + a + a^2 + a^3 + a^4 = (a^5 - 1)/(a - 1)$
Denominator:	$1/a + 1/b = (a + b)/(ab)$
Exponential:	$(e^a)^b = e^{ab}$, $e^a e^b = e^{a+b}$, $a^b = e^{b \log(a)}$
Logarithm:	$\log(ab) = \log(a) + \log(b)$. $\log(a^b) = b \log(a)$
Trig functions:	$\cos^2(x) + \sin^2(x) = 1, \sin(2x) = 2\sin(x)\cos(x), \cos(2x) = \cos^2(x) - \sin^2(x)$
Square roots:	$a^{1/2} = \sqrt{a}$, $a^{-1/2} = 1/\sqrt{a}$

Important functions

Polynomials	x^3+2x^2+3x+1	Exponential	$5e^{3x}$
Rational functions	$(x+1)/(x^3+2x+1)$	Logarithm	$\log(3x)$
Trig functions	$2\cos(3x)$	Inverse trig functions	$\arctan(x)$

Important derivatives

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$f(x)=x^n$	nx^{n-1}	$f(x)=\sin(ax)$	$a\cos(ax)$
$f(x)=e^{ax}$	ae^{ax}	$f(x)=\tan(x)$	$1/\cos^2(x)$
$f(x)=\cos(ax)$	$-a\sin(ax)$	$f(x)=\log(x)$	$1/x$
$f(x)=\arctan(x)$	$1/(1+x^2)$	$f(x)=\sqrt{x}$	$1/(2\sqrt{x})$

Differentiation rules

Addition rule	$(f+g)'=f'+g'$.	Quotient rule	$(f/g)'=(f'g-fg')/g^2$.
Scaling rule	$(cf)'=cf'$.	Chain rule	$(f(g(x))'=f'(g(x))g'(x)$.
Product rule	$(fg)'=f'g+fg'$.	Easy rule	simplify before deriving

Extremal problems

To maximize or minimze f on an interval $[a,b]$, find all critical points inside the interval, evaluate f on the boundary $f(a), f(b)$ and then compare the values to find the global maximum. No second derivative test at the boundary.

Limit examples

$\lim_{x\to 0}\sin(x)/x$	l'Hospital 0/0	$\lim_{x\to 1}(x^2-1)/(x-1)$	heal
$\lim_{x\to 0}(1-\cos(x))/x^2$	l'Hospital 0/0 twice	$\lim_{x\to\infty}\exp(x)/(1+\exp(x))$	l'Hospital
$\lim_{x\to 0}(1/x)/\log(x)$	l'Hospital ∞/∞	$\lim_{x\to 0}(x+1)/(x+5)$	no work necessary

Important things

Summation and rate of change are at the heart of calculus.
The 3 major types of discontinuities are jump, oscillation, infinity.
Dissection and Newton methods are algorithms to find roots.
The fundamental theorem of trigonometry is $\lim_{x\to 0}\sin(x)/x=1$.
The derivative is the limit $Df(x)=[f(x+h)-f(x)]/h$ as $h\to 0$.
The rule $D(1+h)^{x/h}=(1+h)^{x/h}$ leads to $\exp'(x)=\exp(x)$.
If you forget a derivative like of $\arctan(x)$, use the chain rule.

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