INTRODUCTION TO CALCULUS

MATH 1A

Unit 13: Hospital's rule

Lecture

13.1. Hospital's rule allows to compute limits. [1] It is a miracle procedure:

Hospital's rule. If f, g are differentiable and $f(p) = g(p) = 0$ and $g'(p) \neq 0$, then

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f'(x)}{g'(x)} .$$

Lets see how it works in examples:

Example: Lets prove **the fundamental theorem of trigonometry** again:

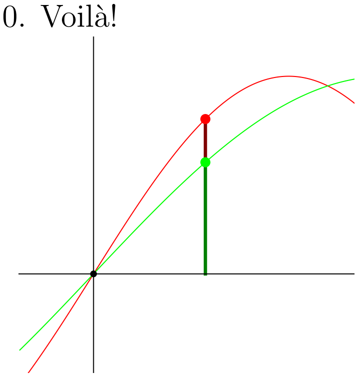
$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1 .$$

In order to apply l'Hospital, we had to know the derivative. Our work to establish the limit was not in vain.

13.2. The proof of the rule is very simple: since $f(p) = g(p) = 0$ we have $Df(p) = (f(p+h) - f(p))/h = f(p+h)/h$ and $Dg(p) = (g(p+h) - g(p))/h = g(p+h)/h$ so that for every $h > 0$ with $g(p+h) \neq 0$ the **quantum l'Hospital rule** holds:

$$\frac{f(p+h)}{g(p+h)} = \frac{Df(p)}{Dg(p)} .$$

Now take the limit $h \to 0$. Voilà!



Example: **Problem.** Find the limit $f(x) = (\exp(2x) - 1)/x$ for $x \to 0$. **Answer.** The rule gives 2.

[1] Also Hôpital. Hospital is is easier to write and remember (bring f to the hospital!)

13.3. Sometimes, we have to administer l'Hospital twice:

If $f(p) = g(p) = f'(p) = g'(p) = 0$ then $\lim_{x\to p} \frac{f(x)}{g(x)} = \lim_{x\to p} \frac{f''(x)}{g''(x)}$ if the limit to the right exists.

Example: **Problem.** Find the limit $f(x) = (\exp(x^2) - 1)/x^2$ for $x \to 0$.

Example: Find the limit $\lim_{x\to 0}(1 - \cos(x))/x^2$. This limit had been pivotal to compute the derivatives of trigonometric functions. **Solution**: differentiation gives

$$\lim_{x\to 0} -\sin(x)/2x \ .$$

Now apply l'Hospital again.

$$\lim_{x\to 0} -\sin(x)/(2x) = \lim_{x\to 0} -\cos(x)/2 = -\frac{1}{2} \ .$$

Example: **Problem:** What do you get if you apply l'Hospital to the limit $[f(x+h) - f(x)]/h$ as $h \to 0$?
Answer: Differentiate both sides with respect to h! And then feel awesome!
Example: Find $\lim_{x\to\infty} x\sin(1/x)$. **Solution.** Write $y = 1/x$ then $\sin(y)/y$. Now we have a limit, where the denominator and nominator both go to zero.

13.4. The case when both sides converge to infinity can be reduced to the 0/0 case by looking at $A = f/g = (1/g(x))/(1/f(x))$ which has the limit $g'(x)/g^2(x)/f'(x)/f^2(x) = g'(x)/f'(x)((1/g)/(1/f))^2 = g'/f'(f^2/g^2) = (g'/f')A^2$, so that $A = f'(p)/g'(p)$. We see:

If $\lim_{x\to p} f(x) = \lim_{x\to p} g(x) = \infty$ for $x \to p$ and $g'(p) \neq 0$, then

$$\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{f'(p)}{g'(p)} \ .$$

Example: What is the limit $\lim_{x\to 0} x^x$? This will provide the best answer to the question **What is 0^0?**
Solution: Because $x^x = e^{x\log(x)}$, it is enough to understand the limit $x\log(x)$ for $x \to 0$.

$$\lim_{x\to 0} \frac{\log(x)}{1/x} \ .$$

Now the limit can be seen as the limit $(1/x)/(-1/x^2) = -x$ which goes to 0. Therefore $\lim_{x\to 0} x^x = 1$. (We assume that $x > 0$ in order to have real values x^x. If we want a function defined everywhere take $|x|^{|x|}$.)
Example: Find the limit $\lim_{x\to 2} \frac{x^2-4x+4}{\sin^2(x-2)}$.
Solution: this is a case where $f(2) = f'(2) = g(2) = g'(2) = 0$ but $g''(0) = 2$. The limit is $f''(2)/g''(2) = 2/2 = 1$.

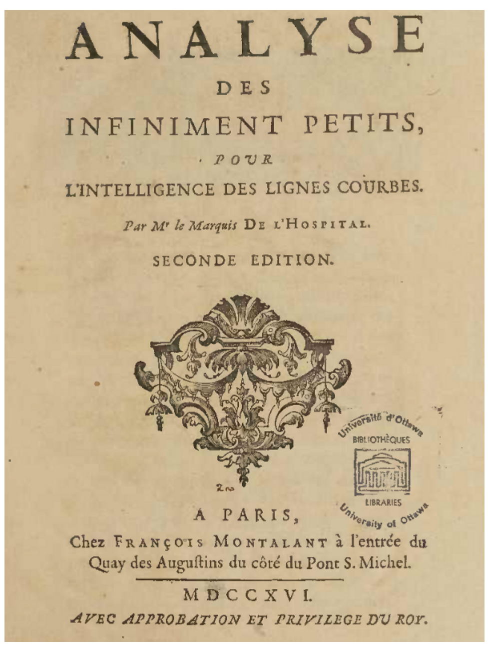
13.5. Hospital's rule always works in calculus situations, where functions are differentiable. The rule can fail if differentiability of f or g fails. Here is an other "rare" example, where one has to think a bit more:

Example: **Deja Vue**: Find $\frac{\sqrt{x^2+1}}{x}$ for $x \to \infty$. L'Hospital gives $x/\sqrt{x^2+1}$ which in terms gives again $\frac{\sqrt{x^2+1}}{x}$. Apply l'Hospital again to get the original function. We got an infinite loop. If the limit is A, then the procedure tells that it is equal to $1/A$. The limit must therefore be 1. This case can be covered easily without going to the hospital: divide both sides by x to get $\sqrt{1+1/x^2}$. Now, we can see the limit 1.

Example: **Trouble?** The limit $\lim_{x\to\infty}(2x+\sin(x))/3x$ is clearly $2/3$ since we can take the sum apart and have $2/3+\sin(x)/(3x)$. Hospital gives $\lim_{x\to\infty}(2+\cos(x))/3$ which has no limit. This is not trouble, since Hospital applies only if the limit $f'(x)$ and $g'(x)$ exists.

History

13.6. The "first calculus book", the world has known was "Analyse des Infiniment Petits pour l"intelligence des Lignes Courbes". It appeared in 1696 and was written by **Guillaume de l'Hospital**, a text if typeset in a modern font would probably fit onto 50-100 pages. [2] It is now clear that the mathematical content in Hospital's book is mostly due to **Johannes Bernoulli**. The book remained the standard for calculus textbooks for a century.


ANALYSE
DES
INFINIMENT PETITS,
POUR
L'INTELLIGENCE DES LIGNES COURBES.
Par M^r le Marquis DE L'HOSPITAL.
SECONDE EDITION.

A PARIS,
Chez FRANÇOIS MONTALANT à l'entrée du Quay des Auguſtins du côté du Pont S. Michel.
MDCCXVI.
AVEC APPROBATION ET PRIVILEGE DU ROY.

[2] Stewart's book with 1200 pages probably contains about 4 million characters, about 12 times more than l'Hospital's book. Modern calculus books also contain more material of course. The OCR text of l'Hospital's book of 200 pages has 300'000 characters.

Homework

Problem 13.1: For the following functions, find the limits as $x \to 0$:
a) $\sin(7x)/5x$
b) $(\exp(16x) - 1)/(\exp(17x) - 1)$
c) $\sin^2(8x)/\sin^2(5x)$
d) $\frac{\tan(4x)}{3x}$
e) $\sin(\sin(11x))/x$.

Problem 13.2: Find the following limits which are indefinite forms ∞/∞ a) $\lim_{x\to 0} \cot(x)/\cot(3x)$.
b) $\lim_{x\to\infty} \frac{3x^2+1}{4x^2+100}$.
c) Find $\lim_{x\to\infty}(x^2 + x - 1)/\sqrt{5x^4 + 1}$.
(**Hint.** To compute the limit faster, Find the limit of $(x^2+x-1)^2/(9x^4+1)$ first, then take the square root of the limit. Apply Hospital several times).

Problem 13.3: Use l'Hospital to compute the following limits $x \to 0$:
a) $\lim_{x\to 0} x/\log|x|$
b) $\log|5x|/\log|x|$.
c) $4\text{sinc}'(x) = 4(\cos(x)x - \sin(x))/x^2$
d) $\log|1 + x|/\log|\log|1 + x||$.
e) $(e^x - 1)/(e^{2x} - 1)$

Problem 13.4: We have seen how to compute limits with healing. Solve this now with l'Hospital at $x \to 1$:
a) $\frac{x^{100}-1}{x^{22}-1}$.
b) $\frac{\tan^2(x-1)}{(\cos(x-1)-1)}$

Problem 13.5: More practice.
a) Find the limit $\lim_{x\to 0} \frac{x}{\tan(6x)}$.
b) Find the limit $\lim_{x\to 5} \frac{x^2-25}{x-5}$
c) Find the limit $\lim_{x\to 0} \frac{1-e^x}{x-x^3}$.
d) Find the limit $\lim_{x\to 0} \frac{\log(1+9x)}{4x}$.
e) Find the limit $\lim_{x\to 1}(x^7 - 1)/(x^3 - 1)$.

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